Matrix

A matrix is a rectangular array of numbers arranged into rows and columns

Eg:

$$A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \qquad B=\begin{bmatrix} 1+x & 2x & -2 \\ 7 & 0 & -1 \\ x^2 & 5 & 4 \end{bmatrix}$$

In the above examples, the horizontal lines of numbers are called *rows* and the vertical lines are called *Columns* of the matrix.

If a matrix has m rows and n columns ($m \neq n$) it is known as *rectangular matrix* and its *order* (or size or dimension) is said to be $m \times n$ (read as m by n).

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each of the number a_{ij} is called an element.

Row Matrix

Any $1 \times n$ matrix is called a row matrix or a row vector

Eg:- [1 2] [1 4 8]

Column Matrix

Any $m \times 1$ matrix is called a column matrix or a column vector

Eg:- $\begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix}$

Square Matrix

A matrix having the same number of rows and columns ($m = n$) is called a square matrix

Eg:- $\begin{bmatrix} 1 & 2 & 4 \\ 16 & 3 & 9 \\ 4 & 16 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Zero Matrix or Null Matrix

A matrix in which every element iz zero is said to be a zero matrix or null matrix

Eg:- $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Diagonal Matrix

A square matrix having non zero entries only on the diagonal is called a diagonal matrix. That is a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

Scalar Matrix

A diagonal matrix with equal non-zero entries on the main diagonal is called a scalar matrix. That is a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$ where k is a number.

Eg:- $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Unit Matrix or Identity Matrix

A diagonal matrix with all the diagonal element equal to unity is called a unit or identity matrix and is denoted by I_n.

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Triangular Matrix

If every element above (or below) the leading diagonal is zero, the matrix is called upper (or lower) triangular matrix.

Eg:- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

Operations on Matrices

I. Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if (i) they are of same order and (ii) each element of A is equal to the corresponding element of B. (i.e. $a_{ij} = b_{ij}$ for all i, j).

If $A = \begin{bmatrix} 1 & 2 & 4 \\ 16 & 3 & 9 \\ 4 & 16 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 \\ 16 & 3 & 9 \\ 4 & 16 & 5 \end{bmatrix}$ then $A = B$.

II. Addition of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order, their sum $C=A+B$ is a defined as the matrix of the same order whose $(i,j)^{th}$ element c_{ij} is obtained by adding the corresponding element of A and B, i.e. $c_{ij}=a_{ij}+b_{ij}$.

If $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 3 & 9 \\ 4 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 4 & 6 & 5 \end{bmatrix}$ then $C = \begin{bmatrix} 1+1 & 2+2 & 4+4 \\ 6+1 & 3+3 & 9+9 \\ 4+4 & 6+1 & 5+5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 8 \\ 7 & 6 & 18 \\ 8 & 7 & 10 \end{bmatrix}$

Properties a matrix addition

1. **Matrix Addition is commutative**

If A and B are nay two matrices of same order then $A+B = B+A$.

Eg.

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then $A+B = \begin{bmatrix} 2+1 & 3+3 \\ -1+2 & 4+4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 1 & 8 \end{bmatrix}$

$B+A = \begin{bmatrix} 1+2 & 3+3 \\ 2-1 & 4+4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 1 & 8 \end{bmatrix}$

Hence $A+B = B+A$.

2. **Matrix addition is associative**

If A, B and C are nay two matrices of same order then $(A+B) + C = A+(B+C)$.

Eg.

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 6 \\ 1 & 8 \end{bmatrix}$ then

$A+B = \begin{bmatrix} 2+1 & 3+3 \\ -1+2 & 4+4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 1 & 8 \end{bmatrix}$, $(A+B)+C = \begin{bmatrix} 3+3 & 6+6 \\ 1+1 & 8+8 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 2 & 16 \end{bmatrix}$

$B+C = \begin{bmatrix} 1+3 & 3+6 \\ 2+1 & 4+8 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 3 & 12 \end{bmatrix}$, $A+(B+C) = \begin{bmatrix} 2+4 & 3+9 \\ -1+3 & 4+12 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 2 & 16 \end{bmatrix}$

Hence $(A+B)+C = A+(B+C)$.

3. Existence of additive identity

Corresponding every $m \times n$ matrix there exist a zero matrix O of same order such that $A + O = O + A = A$.

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $A + O = \begin{bmatrix} 1+0 & 3+0 \\ 2+0 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = A$.

4. Existence of additive inverse

Let $A = [a_{ij}]$ be any matrix of order $m \times n$. Then there exist a matrix B of order $m \times n$, each of whose element is the negative of the corresponding element of A such that $A+B = B+A= O$. Then B is said to be additive inverse of A and is denoted by $-A$.

If $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ then $B = \begin{bmatrix} -1 & -3 \\ 2 & -4 \end{bmatrix}$, $A+B = \begin{bmatrix} 1+(-1) & 3+(-3) \\ -2+2 & 4+(-4) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

III Scalar Multiplication

Let A be a $m \times n$ matrix and k be a scalar. Then $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called the scalar multiple of A by k and is denoted by kA. Thus if $A = [a_{ij}]$ then $k\,A = [ka_{ij}]$.

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then $2\,A = \begin{bmatrix} 1\times 2 & 3\times 2 \\ 2\times 2 & 4\times 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$

Note

1. If A and B are matrices of same order and k be any number, then $k\,(A+B) = kA+kB$.
2. If A and B are matrices of same order and k be any number, then $A-B = A+(-1\times B)$.

IV Matrix Multiplication

The Matrices A and B are said to be conformable for multiplication when the number of columns of A = number of rows of B.

Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. So that the number of columns of A = number of rows of B. Then the product $C = A + B$ is a $m \times p$ matrix where each element c_{ij} of C is obtained by multiplying the elements of i^{th} raw of A with j^{th} column of B and adding the products.

Let $A=[a_{ik}]$ and $B=[b_{kj}]$, i =1, 2, …, m, k =1, 2, …, n and j =1, 2, …, p. Then $C=[c_{ij}]$, where

$$c_{ij} = a_{i1}\,b_{1j} + a_{i2}\,b_{2j} + \ldots + a_{in}\,b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}\,.$$

Eg:

Let $A = \begin{bmatrix} 2 & -1 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$ Then

$$C = AB = \begin{bmatrix} 2\times-1+-1\times2 & 2\times3+-1\times0 & 2\times2+-1\times-1 \\ 4\times-1+-1\times2 & 4\times3+-1\times0 & 4\times2+-1\times-1 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 5 \\ -6 & 12 & 9 \end{bmatrix}$$

Properties of matrix multiplication

1. **Matrix multiplication is associative**

   If A, B and C are three matrices conformable for multiplication, then $A(BC) = (AB)C$

   Eg:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 3 & 7 \\ 1 & 8 & 9 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 7 & 1 \\ 2 & 6 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

Then $AB = \begin{bmatrix} 0+1 & 6+8 & 14+9 \\ 0+4 & 9+32 & 21+36 \end{bmatrix} = \begin{bmatrix} 1 & 14 & 23 \\ 4 & 41 & 57 \end{bmatrix}$

$$(AB)C = \begin{bmatrix} 3+28+23 & 7+84+92 & 1+14+0 \\ 12+82+57 & 28+246+228 & 4+41+0 \end{bmatrix} = \begin{bmatrix} 54 & 183 & 15 \\ 151 & 502 & 45 \end{bmatrix}$$

$$BC = \begin{bmatrix} 0+6+7 & 0+18+28 & 0+3+0 \\ 3+16+9 & 7+48+36 & 1+8+0 \end{bmatrix} = \begin{bmatrix} 13 & 46 & 3 \\ 28 & 91 & 9 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 26+28 & 92+91 & 6+9 \\ 39+112 & 138+364 & 9+36 \end{bmatrix} = \begin{bmatrix} 54 & 183 & 15 \\ 151 & 502 & 45 \end{bmatrix} = (AB)C.$$

2. **Matrix multiplication is distributive**

   If A, B and C are three matrices conformable for multiplication and addition, then $A(B+C) = AB + AC$

Eg:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix},\ B = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

$$B+C = \begin{bmatrix} -2 & 5 & 2 \\ 3 & 4 & 4 \end{bmatrix} \quad A(B+C) = \begin{bmatrix} -1 & 14 & 8 \\ -11 & 16 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 6 & 3 \\ -6 & 12 & 9 \end{bmatrix} \quad AC = \begin{bmatrix} -1 & 8 & 5 \\ -5 & 4 & -5 \end{bmatrix} \quad AB+AC = \begin{bmatrix} -1 & 14 & 8 \\ -11 & 16 & 4 \end{bmatrix}$$

Hence $A(B+C) = AB + AC$

3. Matrix multiplication is non commutative

Let A and B be both wise multiplicative. Then AB is not always equal to BA. In other words matrix multiplication is non commutative.

Eg:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad AB = \begin{bmatrix} -2+2 & 2+-2 \\ -6+4 & 6+-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2+6 & -4+8 \\ 1+-3 & 2+-4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$$

So $AB \neq BA$.

Result

If A and B are two matrices conformable for multiplication and having non zero elements. Then $AB = O$ does not imply that $A = O$ or $B = O$.

Eg:

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 8 \\ 0 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 0\times 6+2\times 0 & 0\times 8+2\times 0 \\ 0\times 6+4\times 0 & 0\times 8+4\times 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

But $A \neq O$ and $B \neq O$.

Transpose of a Matrix

Transpose of a matrix A is the matrix obtained by interchanging the rows and column of A and is denoted by A' or A^T. Thus if A is a $m \times n$ matrix then its transpose is an $n \times m$ matrix.

Eg:

$$\text{If } A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 2 & -1 \end{bmatrix}$$

Result

If A^T and B^T be the transpose of the matrices A and B respectively. Then

(i) $(A^T)^T = A$.
(ii) $(A+B)^T = A^T + B^T$, provided A and B being of the same order
(iii) $(AB)^T = B^T A^T$, provided A and B conformable for multiplication (this is called reversal law of transpose)
(iv) $(kA)^T = k(A^T)$, k is any number.

Eg:

$$\text{If } A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, \; A^T = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2+1 & 1+2 \\ 5+-3 & 3+4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 7 \end{bmatrix} \quad (A+B)^{\mathrm{T}} = \begin{bmatrix} 3 & 2 \\ 3 & 7 \end{bmatrix}$$

$$A^{\mathrm{T}} + B^{\mathrm{T}} = \begin{bmatrix} 2+1 & 5+-3 \\ 1+2 & 3+4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 7 \end{bmatrix} = (A+B)^{\mathrm{T}}$$

$$AB = \begin{bmatrix} 2-3 & 4+4 \\ 5-9 & 10+12 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ -4 & 22 \end{bmatrix}, \ (AB)^{\mathrm{T}} = \begin{bmatrix} -1 & -4 \\ 8 & 22 \end{bmatrix}$$

$$B^{\mathrm{T}} A^{\mathrm{T}} = \begin{bmatrix} 2-3 & 5-9 \\ 4+4 & 10+12 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 22 \end{bmatrix} = (AB)^{\mathrm{T}}.$$

Symmetric matrices and Skew -Symmetric Matrices

A square matrix is said to be *Symmetric Matrix* if it is same as its transpose. That is a square matrix A is symmetric if $A = A^T$.

Eg:

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & c_2 \\ a_3 & c_2 & c_3 \end{bmatrix}$$

A square matrix A is said to be *Skew Symmetric Matrix* if $A^{\mathrm{T}} = - A$.

Note: The elements on the main diagonal of a skew symmetric matrix are all zeros.

Eg:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

Properties

1. If A and B are symmetric matrices. Then AB is symmetric if and only if $AB = BA$.
2. If A be any square matrix then $A + A^{\mathrm{T}}$ is symmetric and $A - A^{\mathrm{T}}$ is skew symmetric
3. Every square matrix can be expressed as the sum of two matrices of which one is symmetric and the other skew symmetric. Let A be any square matrix then $P=½(A+A^{\mathrm{T}})$ is symmetric and $Q = ½(A - A^{\mathrm{T}})$ is skew symmetric matrix. Hence $P+Q =½(A+A^{\mathrm{T}})+ ½(A - A^{\mathrm{T}}) = A$.

Eg: Express $A = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 1 & 0 \\ 6 & 4 & 2 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrices

$A^T = \begin{bmatrix} 0 & 4 & 6 \\ 2 & 1 & 4 \\ 3 & 0 & 2 \end{bmatrix}$ $\quad P = ½(A+A^T) = ½\begin{bmatrix} 0 & 6 & 9 \\ 6 & 2 & 4 \\ 9 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 9/2 \\ 3 & 1 & 2 \\ 9/2 & 2 & 2 \end{bmatrix}$

$Q = ½(A-A^T) = ½\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -3/2 \\ 1 & 0 & -2 \\ 3/2 & 2 & 0 \end{bmatrix}$

$$P + Q = \begin{bmatrix} 0 & 3-1 & \frac{9}{2}-\frac{3}{2} \\ 3+1 & 1-0 & 2-2 \\ \frac{9}{2}+\frac{3}{2} & 2+2 & 0+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 1 & 0 \\ 6 & 4 & 2 \end{bmatrix}.$$

Determinants

To each square matrix $A = [a_{ij}]$ we associate a number called determinant of A, and is dented by det A or $|A|$. Note that the matrices which are not square do not have determinant.

<u>Determinant of Square Matrix of Order One</u>.

The determinant of a 1×1 matrix $A = [a]$ is given by $|A| = a$.

<u>Determinant of Square Matrix of Order Two</u>.

The determinant of a 2×2 matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ then $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

<u>Determinant of Square Matrix of Order Three</u>.

The determinant of a 3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

or

$$|A| = - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

Minor and cofactor of a determinant

Let $A = [a_{ij}]$ be a square matrix of order n. The its determinant $|A|$ is also of order n. If we suppress the I^{th} row and j^{th} column of the determinant, we get a determinant M_{ij}

of order n-1. This determinant M_{ij} is called the minor of the element a_{ij}. The cofactor C_{ij} of a_{ij} is defined as $(-1)^{i+j} M_{ij}$.

Eg:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

M_{11} = minor of the element $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ and the co factor of a_{11} is

$C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

Similarly $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and $C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Now define the value of the determinant $|A|$ of order n as

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

if expanded along the ith row and

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

if expanded along the ith column.

Properties of a determinants

1. The value of a determinant remains unchanged if its row and columns are interchanged.
2. If two rows (or columns) of a determinant are interchanged then the sign of the determinant is changed
3. If any two rows (or column) of a determinant are identical then its value is zero.
4. If each element of a row (or column) of a determinant is multiplied by a constant k, then its value get multiplied by k.
5. If any two rows (or column) of a determinant are proportional then its value is zero.

Adjoint of a matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of the element a_{ij} of the determinant $|A|$, then the matrix $C = [C_{ij}]$ is called the cofactor matrix of A and its transpose is called the adjoint of A and is denoted by adj.A. i.e. adj.$A = C^T$.

Result

Let A be a square matrix of order n, then

$$A \times (\text{adj}.A) = |A|\, I_n = (\text{adj } A)\, A$$

where I_n is the identity matrix of order n.

Singular and Non Singular Matrices

A square matrix A is called a singular matrix if $|A| = 0$ and if $|A| \neq 0$, then A is called non-singular.

Invertible matrix and Inverse of a matrix

Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, where I_n is the identity matrix of order n. The A is said to be invertible and B is called the inverse of A.

Result

1. Inverse of a square matrix if it exists is unique
2. A square matrix is invertible if and only if it is non singular
3. If A is invertible matrix, then its inverse is given by $A^{-1} = \dfrac{\text{adj}.A}{|A|}$
4. If A and B are invertible matrices of the same order then AB is invertible and
$$(AB)^{-1} = B^{-1} A^{-1}.$$

Rank of a Matrix

A non zero matrix is said to have rank k if at least one of its minor is not zero and all minors of order more than k if any are zero.

Orthogonal Matrix

A square matrix A is said to be orthogonal if the product of it is a unit matrix. That is $AA' = A'A = I_n$, then A is orthogonal matrix. If A is orthogonal matrix then $A' = A^{-1}$.

Solution of Simultaneous equations using matrices

Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

This equations are simultaneous equations

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ then the above system of equation can be written as $AX = D$. If A is invertible then we can obtain the solution of the simultaneous equation as $X = A^{-1}D$.

Cramer's Rule

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad A_1 = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix} \quad A_2 = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix} \; A_1 = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

Then Cramer rule gives the solution above system of equation as

$$x = \frac{|A_1|}{|A|}, \; y = \frac{|A_2|}{|A|} \text{ and } z = \frac{|A_3|}{|A|}$$