Operations Research

During World War II, the military management in England called on a team of scientist to study the strategic and technical problem of air and land defence. This team of scientists included physicists, mathematicians, statisticians, engineers, psychologists and many others. The objective was to determine the most effective utilization of limited military resources. This group of scientists forms the first OR team. The name operations research (or operational research) was apparently coined because the team was carrying out research on military operations. Immediately after the war, the success of military team attracted the attention of industrial managers who were seeking solution to their problem. As a result OR soon spread from military to Government, industrial, social and economic planning.

Definition of Operations Research

Operations research is a scientific methods, techniques and tools of providing executive department with a quantitative basis for decision regarding the operations under their control.

Characteristic of Operations Research

The essential characteristic of operations research are

1. OR study the system as a whole. i.e. OR is system (or Executive) Orientation.
2. OR uses interdisciplinary teams
3. The application of Scientific Methods
4. OR cannot cover new problems arises in later stage

Phases of Operations Research

The different phases of the Operations Research approach are

1. Formulation the Problem
2. Construction of a model to represent the system under study
3. Deriving a solution from the model
4. Testing the model and the solution derived from it.
5. Establishing the control over solution.
6. Establishing the solution to work, i.e. implementation.

Models of Operations Research

The main models of Operations Research are

1. Mathematical Programming techniques
2. Inventory Models
3. Allocation Model
4. Transportation Problems
5. Sequencing models
6. Routing models
7. Competitive models
8. Queuing models
9. Dynamic Programming models
10. Simulation techniques
11. Decision theory
12. Replacement models
13. Heuristic models

14. Combined methods

Linear Programming Problem

The mathematical programming problems in general deals with determining optimal allocation of limited resources to meet given objectives. The resources may be materials, men, machines etc. The main branches of mathematical programming are Linear Programming (LPP), Non-Linear Programming (NLPP), Integer Programming (IPP) and Goal Programming. A Linear Programming Problem deals with the optimization (maximization or minimization) of a function of variables known as *Objective function*, subject to a set of linear equations and/or inequations known as restrictions or constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The restrictions may be imposed by different sources such as market demand, production processes and equipment, storage capacity, raw material availability etc. By linearity is meant a mathematical expression (equations) in which the variables do not have powers.

Thus Linear Programming may be defined as a method to obtain an optimum solution to a linear objective function subject to a set of linear constraints as defined above.

The requirement of a Linear Programming Problem

Linear programming problem can be used for optimization problems if the following conditions are satisfied

1. There must be a well defined objective function (in terms of profit, cost or quantity produced) which is to maximized or minimized and which can be expressed as a linear function of decision variables.
2. There must be restrictions on the amount or extent of attainment of the objective and these restrictions must be capable of being expressed as linear equalities or inequalities in terms of variables.
3. There must be alternative course of action. For example, given a product may be processed by two different machines and problem may be as to how much of the product to allocate to which machine.
4. Another necessary requirement is that the decision variables should be interrelated and non-negative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are illogical.
5. Finley the resources must be in limited supply.

General form of a Linear Programming Problem

Optimize $Z = c_1x_1 + c_2x_2 + .. + c_nx_n$.

Subject to

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = (\text{or} \geq \text{or} \leq) b_1$.

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = (\text{or} \geq \text{or} \leq) b_2$

…

$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = (\text{or} \geq \text{or} \leq) b_m.$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

Advantages of a Linear Programming Problem

1. It provides the insight and perspectives into problem environment related with a multi dimensional phenomenon. This generally results in clear picture of the true problem.
2. It makes a scientific and mathematical analysis of the problem situations. It also considers all possible aspects and remedies associates with the problem.
3. It gives an opportunity to the decision maker to formulate his strategies consistent with the constraints and the objectives.
4. It deals with changing situations. Once a plan is arrived through a Linear Programming, it can be revaluated for changing conditions.

Formulation of Linear Programming Problem.

The main steps in formulation of a Linear Programming Problem are

Steps

1. Study the situation and find the key-decision to be made. (i.e. to identify whether the problem is minimization or maximization)
2. Assume symbols for variable quantities or identify the decision variables noticed in step 1.
3. Express the feasible alternative mathematically in terms of variables. Feasible alternative are those which are physically, economically and financially possible.
4. Mention the objective quantity and expressed it as a linear function of the variables.
5. Express the influence factors or restriction (constrains) into mathematical linear functions.

Example –1

A firm produces three products. Three products are processed on three different machines. The time required to manufacture one unit of each of three products and the daily capacity of the tree machine are given in the table below

Machine	Time per unit (minutes)			Machine Capacity
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactures for each product. The profit per unit for product 1, 2 and 3 are Rs.4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the LPP

Step 1: The Key –decision to be made is to determine the number of each product 1, 2 and 3 to be produced so has to maximize profit.

Step 2: The decision variables are x_1, x_2 and x_3, which represent the number of units of product 1, 2 and 3 produced daily.

Step 3: The feasible alternatives are the set of variables x_1, x_2 and x_3, where $x_1, x_2, x_3, \geq 0$.

Step 4: The objective function is Maximize Z= $4x_1+3x_2+6x_3$

Step 5: The constrains are

$$2x_1+3x_2+2x_3 \leq 440$$
$$4x_1+0x_2+3x_3 \leq 470$$
$$2x_1+5x_2+0x_3 \leq 430.$$

Thus the LPP is

Maximize Z= $4x_1+3x_2+6x_3$

Subject to

$$2x_1+3x_2+2x_3 \leq 440$$
$$4x_1+3x_3 \leq 470$$
$$2x_1+5x_2 \leq 430.$$
$$x_1, x_2, x_3, \geq 0.$$

Example –2

An advertising company wishes to plan its advertising strategy in three different media-television, radio and magazines. The purpose of advertising is to reach as large a number of potential customers as possible. Following data has obtained from market survey

	Television	Radio	Magazine I	Magazine II
Cost of advertising units	Rs.30,000	Rs.20,000	Rs.15,000	Rs.10,000
No of potential customers reached per unit	2,00,000	6,00,000	1,50,000	1,00,000
No of female customers reached per unit	1,50,000	4,00,000	70,000	50,000

The company wants to spend mot more than Rs. 450,000 on advertising. Following are the further requirement that must be met:

1. at least 1 million exposures take place among female customers
2. advertising on magazine be limited to Rs.1,50,000
3. at least 3 advertising units be brought on magazine I and 1 units in magazine II and
4. the number of advertising units on television and radio should each be between 5 and 10.

Formulate the LPP

Step 1: The Key –decision to be made is to determine the number of advertising units to be brought in television, radio and magazines so has to reach as large a number of potential customers

Step 2: The decision variables are x_1, x_2, x_3 and x_4, which represent the number of advertising units in television, radio, magazine I and magazine II.

Step 3: The feasible alternatives are the set of variables x_1, x_2, x_3 and x_4, where $x_1, x_2, x_3, x_4 \geq 0$.

Step 4: The objective function is Maximize Z= 10^5 ($2x_1+6x_2 +1.5\, x_3+1\, x_4$)

Step 5: The constrains are

$$30{,}000x_1+20{,}000x_2 +15{,}000x_3 +10{,}000x_4 \leq 4{,}50{,}000$$
$$150{,}000x_1+400{,}000x_2 +70{,}000x_3 +50{,}000x_4 \geq 1{,}00{,}000$$
$$15{,}000x_3 +10{,}000x_4 \leq 1{,}50{,}000$$
$$x_3 \geq 3$$
$$x_4 \geq 2$$
$$5 \leq x_1 \leq 10$$
$$5 \leq x_2 \leq 10$$

Thus the LPP is

$$\text{Maximize } Z= 10^5\, (2x_1+6x_2 +1.5\, x_3+1\, x_4)$$

Subject to

$$30x_1+20x_2 +15x_3 +10x_4 \leq 450$$
$$15x_1+40x_2 +7x_3 +x_4 \geq 100$$
$$15x_3 +10x_4 \leq 150$$
$$x_3 \geq 3$$
$$x_4 \geq 2$$
$$x_1 \geq 5$$
$$x_2 \geq 5$$
$$x_1 \leq 10$$
$$x_2 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution of Linear Programming Problem

In general there are two methods to solve the Linear Programming Problem. They are graphical methods if there are only two decision variables and the simplex method. Now we consider some definition related to this

Feasible Solution: A feasible solution to a Linear Programming Problem is the set of the values of the variables which satisfy all the.

Basic feasible solutions: A feasible solution is called a basic feasible solution if it has no more than *m* positive (non-zero) x_j. In other words, it is a basic solution which also satisfies the non-negativity condition.

Optimal Solution: A basic feasible solution to a Linear Programming Problem is said to be optimum solution if it optimizes the objective function of the problem.

Slack Variables: If the constrains has a ≤ then in order to make it an equality we have to add some variable to the left hand side of the constraints. This variables are called slack variables.

Suruplus Variables: If the constrains has a sign ≥ then in order to make it an equality we have to subtract some variable to the left hand side of the constraints. This variables are called surplus variables.

Graphical Method of Solution to Linear Programming Problem

Linear Programming Problem involving two variables can be solved by Graphical method. Since the solution has to satisfies the non negative conditions the value of the variable x_1 and x_2 can lie only in the first quadrant. The main steps for solving a LPP by graphic method are

1. Formulate the problem into a Linear Programming Problem
2. Each inequality in the constrains may be written as equality
3. Draw straight lines corresponding to the equations obtained in step 2. So there will be many straight line as the constraints are.
4. Identify the feasible region. Feasible region is the area which satisfies all the constrains simultaneously
5. The vertices of the feasible solutions are to be located and their co ordinates are to be measured.
6. Calculate the value of the objective function at each vertex.
7. The solution is the co-ordinate of the vertex, which optimizes the objective function, and the corresponding value of the objective function is the optimum value.

Example

Solve the following LPP

$$\text{Maximize } Z = 3x_1+5x_2$$

Subject to

$$2x_1+6x_2 \leq 36$$
$$3x_1+2x_2 \leq 24$$
$$x_1, x_2, \geq 0.$$

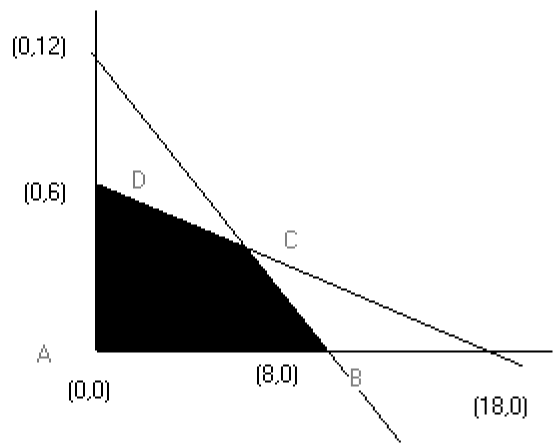
Change the constrains to equality we have

$$2x_1+6x_2 = 36 \qquad (1)$$
$$3x_1+2x_2 = 24 \qquad (2)$$

Now consider equation (1) ,when $x_1 = 0$, $x_2 = 6$ and when $x_2 = 0$, $x_1 = 18$, so the two point in line (1) are (0,6) and (18,0). Similarly for equation (2) we have two points (0,12) and (8,0). Plotting this we get the straight line graphs as

Vertex	Function value of Z
A=(0,0)	0
B=(8,0)	24
C=(36/7,60/7)	58.2857
D=(0,6)	30

So the optimum solution is (36/7,60/7) and the optimum value is 58.2857.



Duality in Linear Programming Problem

For every Linear Programming Problem there is a related unique another Linear Programming Problem involving the same data and is called the dual programme. The given original programming problem is called primal programme. A solution to the dual programme may be found in a manner similar to that used for the primal. The two programmes have very closely related properties so that optimal solution of the dual gives the complete information about the optimal solution of the primal and vice-versa.

Duality is extremely important and interesting feature of linear programming. The various useful aspects of this properties are

1. If the primal problem contains a large number of constrains and a smaller number of variables., the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence it offers an advantages in many application
2. It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of problem. This is termed as post optimality or sensitivity analysis.
3. duality in linear programming has certain far reaching consequences of economic nature.
4. Calculation of the dual cheeks the accuracy of the primal solution
5. duality in linear programming shows that each linear programme is equivalent to a two-person-zero-sum game. This indicates that fairly close relationships exists between linear programming and theory of games.

Dual - Primal problems

The general linear programming problem has the form

maximize $Z = c_1x_1 + c_2x_2 + .. + c_nx_n$.

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1.$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = \le b_m.$$
$$x_1, x_2, \ldots, x_n \ge 0.$$

Then its dual is

Minimise $W = b_1y_1 + b_2y_2 + .. + b_my_m$.

Subject to

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1$$
$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2$$
$$\ldots$$
$$a_{1n}y_1 + a_{2n}y_2 + .. + a_{nm}y_m \ge c_m.$$
$$y_1, y_2, \ldots, y_m \ge 0.$$

Comparing the two problem we have the following points

1. If the primal contains n variables and m constraints, the dual will contains m variables and n constraints.
2. The maximization problem in the primal become the minimization problem in the dual and vice versa.
3. The maximization problem has ($\le$) constraints while the minimization problem has ($\ge$) constraints.
4. The constants $c_1, c_2, \ldots, c_n$ in the objective function of the primal appear in the constraints of the duals.
5. The constants $b_1, b_2, \ldots, b_n$ in the constraints of the primal appear in the objective function of the duals.
6. The variables in both problem are non negative.

The constraint relationship of the primal and dual can be represented in a single table as shown below

	x_1	x_2	x_3	$\ldots$	x_n	
y_1	a_{11}	a_{12}	a_{13}	$\ldots$	a_{1n}	$\le b_1$
y_2	a_{21}	a_{22}	a_{23}	$\ldots$	a_{2n}	$\le b_2$
y_3	a_{31}	a_{32}	a_{33}	$\ldots$	A_{3n}	$\le b_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	a_{m3}	$\ldots$	a_{mn}	$\le b_m$
	$\ge c_1$	$\ge c_2$	$\ge c_3$	$\ldots$	$\ge c_n$	

Example-1

Construct the dual to the primal problem

Maximize $Z = 3x_1 + 5x_2$

Subject to

$2x_1 + 6x_2 \le 50$
$3x_1 + 2x_2 \le 35$
$5x_1 - 3x_2 \le 10.$
$x_1, x_2, \ge 0.$

Solution

Let y_1, y_2 and y_3 be the corresponding dual variables, then the dual problem is given by

Minimize $W = 50y_1 + 35y_2 + 10y_3$

Subject to

$2y_1 + 3y_2 + 5y_3 \ge 3$
$6y_1 + 2y_2 - 3y_3 \ge 5$
$y_1, y_2, y_3, \ge 0.$

Example-2

Construct the dual to the primal problem

Minimize $Z = 3x_1 + 5x_2$

Subject to

$2x_1 + 6x_2 \ge 36$
$3x_1 + 2x_2 \ge 24$
$x_1, x_2, \ge 0.$

Solution

Let y_1 and y_2 be the corresponding dual variables, then the dual problem is given by

Maximize $W = 36y_1 + 24y_2$

Subject to

$2y_1 + 3y_2 \le 3$
$6y_1 + 2y_2 \le 5$
$y_1, y_2 \ge 0.$

LINEAR PROGRAMMING PROBLEM II.
(SIMPLEX METHOD AND DUALITY METHOD)

Simplex method was originally developed by G.B.Dantzig, an American mathematician.

Simplex method is a Linear Programming technique in which we start with a certain solution which is feasible. We improve this solution in a number of consecutive stages until we arrive at an optimal solution.

For arriving at the solution of LPP by this method, the constraints and the objective function are presented in a table known as simplex table. Then following a set procedure and rules, the optimal solution is obtained making step by step improvement.

Thus Simplex method is an iterative (step by step) procedure in which we proceed in a systematic steps from an initial Basic Feasible Solution to another Basic Feasible Solution and finally, in a finite number of steps to an optimal basic feasible solution, in such a way that the value of the objective function at each step is better (or at least not worse) than that at the preceding steps. In other words the simplex algorithm consists of the following main steps

(1) Find a trial Basic Feasible Solution of the Linear Programming Problem.
(2) Test whether it is an optimal solution or not.
(3) If not optimal, improve the first trial Basic Feasible Solution by a set of rules. That is determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value
(4) Repeat the steps (2) and (3) till an optimal solution is obtained.

How to construct a simplex table?

Simplex table consists of rows and columns. If there are 'm' original variables and 'n' introduced variables, then there will be 3+m+n columns in the simplex table. (Introduced variables are slack, surplus or artificial variables).

First column (B) contains the basic variables. Second column (C) shows the coefficient of the basic variables in the objective function. Third column (x_B) gives the values of basic variables. Each of next 'm+n' columns contain coefficient of variables in the constraints, when they are converted into equations.

Basic (B)

The variables whose values are not restricted to zero in the current basic solution, are listed in one column of the simplex table known as Basis (B).

Basic variables

The variables which are listed in the basis are called basic varibles and others are known as non-basic variables.

Vector

Any column or row of a simplex table is called a vector. So we have X_1 – vector, X_2 – vector etc.

In a simplex table, there is a vector associated with every variable. The vectors associated with the basic variables are unit vectors.

Unit vector

A vector with one element 1 and all other elements zero, is a unit vector.

Eg: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ are unit vectors.

Net Evaluation (Δ_j)

Δ_j is the net profit or loss if one unit of the variable in the respective column is introduced. That is, Δ_j shows what is the profit (or loss) if one unit of x_j is introduced. The row containing Δ_j values is called net evaluation row or index row.

$\Delta_j = z_j - c_j$

where c_j is the coefficient of x_j variables in the objective function and z_j is the sum of the products of coefficients of basic variables in the objective function and the vector x_j.

Minimum Ratio

Minimum ratio is the lowest non negative ratio in the replacing ratio column. The replacing ratio column (θ) contains values obtained by dividing each element in x_B column (the column showing the values of the basic variable) by the corresponding elements in the incoming vector.

Key Column (incoming vector)

The column which has highest negative Δ_j in a maximisation problem or the highest positive Δ_j in a minimisation problem, is called incoming vector.

Key row (outgoing vector)

The row which relates to the minimum ratio, is the outgoing vector.

Key element

Key element is that element of the simplex table which lies both in the key row and key column.

Iteration

Iteration means step by step process followed in simplex method to move from one basic feasible solution to another.

Computational procedure of simplex method (Simplex Algorithm)

Step1: Formulate the problem into a LPP

Step2: Convert the constraints into equations by introducing the non-negative slack variables or surplus variables wherever necessary.

Step3: Construct starting simplex table.

Step4: Conduct the test of optimality.

This is done by computing net evaluation $\Delta_j = Z_j - C_j$

The solution under test is not optimal if at least one Δ_j is negative for maximisation case. If at least on Δ_j is positive, the solution is not optimal for minimisation case. Otherwise the solution is optimal.

If solution under test is not optimal, we must proceed to the next step.

Step5: Find incoming and outgoing vectors.

The incoming vector corresponds to highest negative Δ_j for maximisation cases and highest positive Δ_j for minimisation cases. Outgoing vector corresponds to minimum ratio.

Step6: The element which is at the intersection of minimum ratio arrow (⟵) and incoming vector arrow (↑) is called the key element. We mark this element in

There should be 1 at the positon of key element. If it is not 1 then divide all the elements of the row, containing key element, by the key element. Then add appropriate multiples of the corresponding elements of this changed row to the elements of all other rows. Now obtain the next simplex table with the changes. Improved basic feasible solution can be readout from the simplex table. The solution is obtained by reading B – column and x_B column together.

Step7: Now test the above improved B. F. S. for optimality as in step 4.

If this solution is not optimal then repeat steps (5) and (6) until an optimal solution is finally obtained.

Artificial variable

Artificial variables are fictitious variables. They are incorporated only for computational purposes. They have no physical meaning. Artificial variables are introduced when the constraints are of the type ≥ or =.

Big – M Method

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent. When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

Big M method is a modified simplex method for solving a LPP when a high penalty cost (or profit) M has been assigned to the artificial variable in the objective function.

When artificial variables are introduced, we include these artificial variables in the basis (B) first. These artificial variables are driven out in the first iteration. For this purpose, we assign a very large M to teach artificial variables as coefficient in the objective function, The quantity 'M' is known as penalty. In maximisation cases –M and in minimisation cases +M are assigned to the artificial variables astheir coefficients in objective functions.

Big M method can be applied to minimisation problems as well as maximisation problems.

Simplex Method with 'greater-than-equal-to' (ε) and equality (=) constraints

The LP problem, with 'greater-than-equal-to' (ε) and equality (=) constraints, is transformed to its standard form in the following way.

1. One 'artificial variable' is added to each of the 'greater-than-equal-to' (ε) and equality (=) constraints to ensure an initial basic feasible solution.
2. Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient *M* for maximization (minimization) problem.
3. Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.
4. If there are more than one artificial variable, step 3 is repeated for all the artificial variables one by one.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the

inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.

2. Convert each inequality constraint to standard form (If constraint *i* is a ≤ constraint, we add a slack variable *si*; and if constraint *i* is a ≥ constraint, we subtract an excess variable *ei*).
3. Add an artificial variable *ai* to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction *ai* ≥ 0.
4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) M*ai* to the objective function. If the LP is a max problem, add (for each artificial variable) -M*ai* to the objective function.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Duality in Linear Programming Problem

For every Linear Programming Problem there is a related unique another Linear Programming Problem involving the same data and is called the dual programme. The given original programming problem is called primal programme. A solution to the dual programme may be found in a manner similar to that used for the primal. The two programmes have very closely related properties so that optimal solution of the dual gives the complete information about the optimal solution of the primal and vice-versa.

Duality is extremely important and interesting feature of linear programming. The various useful aspects of this properties are

6. If the primal problem contains a large number of constrains and a smaller number of variables., the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence it offers an advantages in many application
7. It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of problem. This is termed as post optimality or sensitivity analysis.
8. duality in linear programming has certain far reaching consequences of economic nature.
9. Calculation of the dual cheeks the accuracy of the primal solution
10. duality in linear programming shows that each linear programme is equivalent to a two-person-zero-sum game. This indicates that fairly close relationships exists between linear programming and theory of games.

Dual - Primal problems

Associated with any LP is another LP called the ***dual***. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min problem and vice versa.

The general linear programming problem has the form
maximize $Z = c_1x_1 + c_2x_2 + .. + c_nx_n$.

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1.$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = \leq b_m.$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

Then its dual is

Minimise $W = b_1y_1 + b_2y_2 + .. + b_my_m$.

Subject to

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2$$
$$\ldots$$
$$a_{1n}y_1 + a_{2n}y_2 + .. + a_{nm}y_m \geq c_m.$$
$$y_1, y_2, \ldots, y_m \geq 0.$$

Comparing the two problem we have the following points

7. If the primal contains n variables and m constraints, the dual will contains m variables and n constraints.
8. The maximization problem in the primal become the minimization problem in the dual and vice versa.
9. The maximization problem has (≤) constraints while the minimization problem has (≥) constraints.
10. The constants $c_1, c_2, \ldots, c_n$ in the objective function of the primal appear in the constraints of the duals.
11. The constants $b_1, b_2, \ldots, b_n$ in the constraints of the primal appear in the objective function of the duals.
12. The variables in both problem are non negative.

The constraint relationship of the primal and dual can be represented in a single table as shown below

	x_1	x_2	x_3	…	x_n	
y_1	a_{11}	a_{12}	a_{13}	…	a_{1n}	$\leq b_1$
y_2	a_{21}	a_{22}	a_{23}	…	a_{2n}	$\leq b_2$
y_3	a_{31}	a_{32}	a_{33}	…	A_{3n}	$\leq b_3$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leq b_m$
	$\geq c_1$	$\geq c_2$	$\geq c_3$	$\cdots$	$\geq c_n$	

Example-1

Construct the dual to the primal problem

Maximize $Z = 3x_1 + 5x_2$

Subject to

$$2x_1+6x_2 \leq 50$$
$$3x_1+2x_2 \leq 35$$
$$5x_1-3x_2 \leq 10.$$
$$x_1, x_2, \geq 0.$$

Solution

Let y_1, y_2 and y_3 be the corresponding dual variables, then the dual problem is given by

Minimize $W = 50y_1+35y_2 +10 y_3$

Subject to

$$2y_1+3y_2 +5y_3 \geq 3$$
$$6y_1+2y_2 -3y_3 \geq 5$$
$$y_1, y_2, y_3, \geq 0.$$

Example-2

Construct the dual to the primal problem

Minimize $Z = 3x_1 + 5x_2$

Subject to

$$2x_1+6x_2 \geq 36$$
$$3x_1+2x_2 \geq 24$$
$$x_1, x_2, \geq 0.$$

Solution

Let y_1 and y_2 be the corresponding dual variables, then the dual problem is given by

Maximize $W = 36y_1+24 y_2$

Subject to

$$2y_1+3y_2 \leq 3$$
$$6y_1+2y_2 \leq 5$$
$$y_1, y_2 \geq 0.$$

Finding the Dual of a Nonnormal Max Problem

1. If the ith primal constraint is a $\geq$ constraint, the corresponding dual variable yi must satisfy yi ≤ 0

2. If the *i*th primal constraint is an equality constraint, the dual variable *yi* is now unrestricted in sign (urs).
3. If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

Finding the Dual of a Nonnormal Min Problem

1. If the *i*th primal constraint is a ≤ constraint, the corresponding dual variable *xi* must satisfy xi ≤ 0
2. If the *i*th primal constraint is an equality constraint, the dual variable *xi* is now urs.
3. If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

The Dual Theorem

The primal and dual have equal optimal objective function values (if the problems have optimal solutions).
Weak duality implies that if for any feasible solution to the primal and an feasible solution to the dual, the *w*-value for the feasible dual solution will be at least as large as the *z*-value for the feasible primal solution
$z \leq w$.

Consequences

Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.

If the primal is unbounded, then the dual problem is infeasible.

If the dual is unbounded, then the primal is infeasible.

Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as ***resource shadow prices***.