Mathematical Expectation-Limitations

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Expectation Limitations

- Let $x_i = 2^i$ and $f(x_i) = \frac{1}{2^i}$. i=1,2,…
- Here E(X) = $\sum 2^i \frac{1}{2^i} = \sum i = \infty$. So E(X) does not exist.
- Let $x_i = \sum(-1)^{i+1}\frac{m^i}{i}$, m>1 and $f(x_i) = \frac{m-1}{m^i}$ i=1,2,…
- Here E(X) = $\sum(-1)^i\frac{m^i}{i}\frac{m-1}{m^i}$ = (m-1) $\sum(-1)^i\frac{1}{i}$.
- Here the series convergent but not absolutely convergent. If we rearrange terms taking the positive terms together and the negative terms together, the sum become which is ∞ - ∞ indeterminate. So E(X) does not exist.
- Let $f(x) = \frac{1}{\pi(1+x^2)}$, - ∞ <x< ∞ then E(X) = $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{(1+x^2)}dx$
- Here the RHS is not convergent so we say that E(X) does not exist.

MOMENT GENERATING FUNCTION

The moment generating function (mgf) for a random variable X is defined as:

$M_X(t) = E(e^{tx})$, $-\infty < t < \infty$, provided expectation exists.

$=\sum_x e^{tx} f(x$, if X is discrete

$=\int e^{tx} f(x)\, dx$, if X is continuous

Proposition 6.3.3. *The k-th moment μ_k of a random variable with the moment-generating function $m_Y(t)$ is given by*

$$\mu_k = \frac{d^k}{dt^k} m_Y(0)|_{t=0}, \qquad (6.3.2)$$

as long as m_Y is defined for t in some neighborhood of 0.

Limitations MGF

- Even when a random variable has no moments, it may have a moment generating function. For example let $f(x) = \frac{1}{x(x+1)}$, x=1,2,3,… and 0 otherwise
- Now E(X^r) = $\sum_1^\infty \frac{x^r}{x(x+1)}$ = $\sum_1^\infty \frac{x^{r-1}}{(x+1)}$ = $\frac{1}{2} + \frac{2^{r-1}}{3} + \cdots$. Which is divergent so E(X^r) does not exist. So the density has no moments
- But $M_x(t) = E(e^{tx})$= $\sum \frac{e^{tx}}{x(x+1)}$ = $\sum_1^\infty \frac{z^x}{x(x+1)}$ where z = e^t

$$= z\left[1-\frac{1}{2}\right] + z^2\left[\frac{1}{2}-\frac{1}{3}\right] + z^3\left[\frac{1}{3}-\frac{1}{4}\right] + \ldots$$

$$= \left[z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right] - \frac{1}{z}\left[\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right]$$

$$= -\log(1-z) + 1 + \frac{1}{z}\log(1-z) \quad \text{for } |z| < 1$$

$$= 1 + \left(\frac{1}{z} - 1\right)\log(1-z) = 1 + (e^{-t}-1)\log(1-e^t).$$

- The condition $|z| < 1$ means e^t<1 ie t<0. If t=0, $M_x(t)$ =1 and if t>0, $M_x(t)$ does not exist. So even though no moments exist the moment generating function exists at least for t≤0.

- A random variable can have all moments and the moments generating function but the m.g.f. may not generate the moments. For example, let
- $f(x) = \frac{e^{-1}}{x!}$ where $x = 2^m$, m=0,1,2,… is the pdf of X.
- E(X^r) = $\sum_{m=0}^{\infty} \frac{(2^m)^r e^{-1}}{m!}$ = $\sum_{m=0}^{\infty} \frac{(2^r)^m e^{-1}}{m!}$ = $e^{2^r - 1}$. So X has moments of all order.
- But $M_x(t) = E(e^{tx})$= $\sum_{m=0}^{\infty} \frac{e^{2^m t} e^{-1}}{m!}$ = $e^{-1} \sum_{m=0}^{\infty} \frac{e^{2^m t}}{m!}$
- This infinite series converges for t≤0 and diverges for t>0. So the m.g.f exist for all t≤0. But cannot be differentiable and cannot be expanded as power series in t. So it cannot generate the moments

- All random variables need not have moment generating functions. The following are examples of such variables,
- (a) $f(x) = \frac{6}{\pi^2 x^2}, \mathrm{x} = 1, 2, 3, 4, \ldots$
- Then $M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$
- This series is divergent for all values of t and so m.g.f does not exist
- (b) $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$
- then $M_x(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{tx}}{(1+x^2)} dx$
- This integral is divergent and so m.g.f does not exist.

Characteristic Function

For a random variable X, the characteristic function, $\Phi x(t)$ is defined as : $\Phi x(t) = E(e^{itX})$, where t is a real number and $i = \sqrt{-1}$. If $f(x)$ is a pdf of a r.v X then $\Phi x(t) = \sum_{-\infty}^{\infty} e^{itx} f(x)$ if X is discrete and $\Phi x(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$ if X is continuous.

Properties of characteristic Function

1. $|\Phi x(t)| \leq 1$ for all values of t.
2. $|\Phi x(0)| = 1$
3. $|\Phi x(-t)| = |\Phi x(t)|$.

Example

- 1. Find the characteristic function of X, if $f(x) = \frac{e^{-m}m^x}{x!}$ where $x = 0,1,2,\ldots$
- Then $\phi_x(t) = E(e^{itx})= \sum_{x=0}^{\infty} \frac{e^{itx}e^{-m}m^x}{x!} = e^{m(e^{it}-1)}$
- 2. Find the characteristic function of X, if $f(x) = ae^{-a}$ a>0, x>0
- $\phi_x(t) = E\left(e^{itx}\right) = a\int_0^{\infty} e^{itx}e^{-ax}dx$
- $= a\int_0^{\infty} e^{-(a-it)x}dx = \frac{a}{a-it} = \left(1 - \frac{it}{a}\right)^{-1}.$

- For Quarries

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