5. Uniform distribution

A continuous random variable X is said to follow a continuous uniform distribution over an interval [a, b] if its probability density function of is given by

$$f(x) = \frac{1}{b-a}, \ a \le x \le b$$

The cumulative distribution function is given by

$$F(x) = \frac{x-a}{b-a}, \ a \le x \le b$$

The uniform distribution with support $[a, b]$ is denoted by $U(a, b)$. This distribution is also called the *rectangular* distribution because of the rectangular shape of its pdf.

1. **Moments.**

$$\mu_r' = \int_a^b \frac{x^r dx}{b-a} = \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b = \frac{1}{b-a}\left[\frac{b^{r+1}-a^{r+1}}{r+1}\right]$$

$$\text{Mean} = \mu_1' == \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \frac{b+a}{2}$$

$$\mu_2' = \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] = \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = =\mu_2 = \mu_2' - (\mu_1')^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

2. **Moment generating function**

$$M_x(t) = \int_a^b \frac{e^{tx} dx}{b-a} = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{e^{bt}-e^{at}}{t(b-a)}.$$

6. Exponential distribution

A continuous random variable X is said to follow a exponential distribution with parameter λ it its probability density function of a is given by

$$f(x) = \lambda e^{-\lambda x}, \ x \ge 0.$$

The cumulative distribution function is given by

$$F(x) = 1 - e^{-\lambda x}, \ x > 0$$

$$f(x) = \lambda e^{-\lambda(x-\mu)},\ x \geq \mu.$$

If we take $\lambda = \frac{1}{b}$ the density become

$$f(x) = \frac{1}{b} e^{-\frac{x}{b}},\ x \geq 0,\ b > 0.$$

1. **Moments.**

$\mu_r' = \int_0^\infty x^r \lambda e^{-\lambda} \ dx = \left[\frac{\lambda\Gamma(r+1)}{-\lambda^{r+1}}\right]_0^\infty = \frac{\Gamma(r+1)}{\lambda^r}$

Mean = $\mu_1' = \frac{\Gamma(1+1)}{\lambda} = \frac{1}{\lambda}$

$\mu_2' = \frac{\Gamma(2+1)}{\lambda^2} = \frac{2}{\lambda^2}$

Variance = =$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$.

2. **Moment generating function**

$M_x(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx = \int_0^\infty \lambda e^{-x(\lambda - t)}\, dx = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$.

3. **Reproductive property**

If X_i (i=1, 2, …, k) are identically independently distributed exponential random variate with parameters λ. Then $S_k = \sum_{i=1}^{k} X_i$ has a gamma distribution with parameters (k, 1/λ). (This property is called additive or reproductive property).

Given $X_i \sim E(\lambda)$ So $M_{x_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$.

$$M_{S_k}(t) = \prod_{i=1}^{k} M_{x_i}(t)$$

$$= \prod_{i=1}^{k} \left(1 - \frac{t}{\lambda}\right)^{-1} = \left(1 - \frac{t}{\lambda}\right)^{-k}$$

Which is the mgf of gamma distribution (k, λ). So $S_k = \sum_{i=1}^{k} X_i$ has a gamma distribution with parameters (k, 1/λ).

Result

The exponential distribution has the memoryless (forgetfulness) property.

A exponential random variable X has the memoryless property if for all nonnegative integers s and t,

$$P(X \geq s+t|X \geq t) = P(X \geq s)$$

Proof:

The probability mass function for a exponential random variable X is

$$f(x) = \lambda e^{-\lambda x},\ x>0.$$

Then

$$P(X \geq x) = e^{-\lambda}$$

$$P(X \geq s+t|X \geq t) = \frac{P(X \geq s+t \cap x \geq t)}{P(X \geq t)} = \frac{P(X \geq s+t)}{P(X \geq t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X \geq s)$$

which proves the memoryless property.

7. Gamma distribution

A continuous random variable X is said to follow a gamma distribution with parameter a and b if its probability density function is given by

$$f(x) = \frac{1}{b^a \Gamma(a)}\ x^{a-1} e^{-x/b},\ x \geq 0.$$

If we take b=1, then we get one parameter gamma distribution or standard gamma distribution. That is in this case the pdf is

$$f(x) = \frac{1}{\Gamma(a)}\, x^{a-1} e^{-x},\ x \geq 0.$$

Mean	ab
Variance	ab^2
Mode	$b(a-1)$, a>1
Coefficient of variation	$\frac{1}{\sqrt{a}}$
Coefficient of Skewness	$\frac{2}{\sqrt{a}}$
Coefficient of Kurtosis	$3 + \frac{6}{a}$
Moment generating function	$(1 - bt)^{-a},\ t < \frac{1}{b}$.

$$f(x) = \frac{1}{b^a \Gamma(a)}\, x^{a-1} e^{-x/b},\ x \geq 0.$$

1. **Moments.**

$$\mu_r' = \int_0^\infty x^r \frac{1}{b^a \Gamma(a)}\, x^{a-1} e^{-x/b}\, dx = \int_0^\infty \frac{1}{b^a \Gamma(a)}\, x^{a+r-1} e^{-x/b}\, dx$$

$$= \frac{1}{b^a \Gamma(a)} \frac{\Gamma(a+r)}{\left(\frac{1}{b}\right)^{a+r}} = \frac{b^r \Gamma(a+r)}{\Gamma(a)}$$

$$\text{Mean} = \mu_1' = \frac{b^1 \Gamma(a+1)}{\Gamma(a)} = ab.$$

$$\mu_2' = \frac{b^2 \Gamma(a+2)}{\Gamma(a)} = a(a+1)b^2.$$

Variance = $=\mu_2 = \mu_2' - (\mu_1')^2 = a(a+1)b^2 - (ab)^2 = ab^2$.

2 Moment generating function

$$M_x(t) = \int_0^\infty e^{tx} \frac{1}{b^a\Gamma(a)} x^{a-1}e^{-x/b}\, dx$$

$$= \int_0^\infty \frac{1}{b^a\Gamma(a)} x^{a-1}e^{-x(\frac{1}{b}-t)}\, dx = \frac{1}{b^a\Gamma(a)} \frac{\Gamma(a)}{\left(\frac{1}{b}-t\right)^a} = (1-bt)^{-a}.$$

3 Reproductive property

If X_i (i=1, 2, …, k) are identically independently distributed gamma random variate with parameters (a_i, b). Then $S_k = \sum_{i=1}^{k} X_i$ has a gamma distribution with parameters $\left(\sum_{i=1}^{k} a_i, b\right)$.

Given $X_i \sim G(a_i, b)$ So $M_{x_i}(t) = (1-bt)^{-a_i}$.

$$M_{S_k}(t) = \prod_{i=1}^{k} M_{x_i}(t)$$

$$= \prod_{i=1}^{k} (1-bt)^{-a_i} = (1-bt)^{-\sum a_i}$$

Hence $S_k = \sum_{i=1}^{k} X_i$ has a gamma distribution with parameters $\left(\sum_{i=1}^{k} a_i, b\right)$.

8. Beta distribution of first kind

A continuous random variable X is said to follow a beta distribution with parameter a and b if its probability density function is given by

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1},\ 0 \le x \le 1.$$

Where $$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Moments

$$\mu_r' = \int_0^1 x^r \frac{1}{B(a,b)}\, x^{a-1}(1-x)^{b-1}\,dx$$

$$= \int_0^1 \frac{1}{B(a,b)}\, x^{a+r-1}(1-x)^{b-1}\,dx = \frac{b(a+r,b)}{B(a,b)}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+r)\Gamma(b)}{\Gamma(a+b+r)} = \frac{\Gamma(a+r)\Gamma(a+b)}{\Gamma(\mathrm{a})\Gamma(a+b+r)}.$$

Mean	$\frac{a}{a+b}$
Variance	$\frac{ab}{(a+b+1)(a+b)^2}$
Mode	$\frac{a-1}{a+b-2}$
Mean Deviation	$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{2a^a b^b}{(a+b)^{(a+b+1)}}$
Coefficient of variation	$\frac{\sqrt{b}}{\sqrt{a(a+b+1)}}$
Coefficient of Skewness	$\frac{2(b-a)\sqrt{(a+b+1)}}{(a+b+2)\sqrt{ab}}$
Coefficient of Kurtosis	$\frac{3(a+b+1)[2(a+b)^2+ab(a+b-6)}{ab(a+b+2)(a+b+3)}$
Characteristic function	$\frac{\Gamma(a+b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)(it)^2}{\Gamma(a+b+k)\Gamma(k+1)}.$
Moments about the Origin:	$E(X^k) = \prod_{i=0}^{k-1}\frac{a+i}{a+b+i}$, k=1,2,…

Beta distribution of second kind

A continuous random variable X is said to follow a beta distribution of second kind with parameter a and b if its probability density function is given by

$$f(x) = \frac{1}{B(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}}, \; 0 \leq x \leq \infty.$$

where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Mean $= \frac{a}{b-1}$

Variance $= \frac{a(a+b-1)}{(b-2)(b-1)^2}$.

Result

If X follows a beta distribution of first kind with parameter a and b. Then $Y = \frac{X}{1-X}$ follow a beta distribution of second kind with parameter a and b.

Proof

Given X follows a beta distribution of first kind with parameter a and b. So

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \; 0 \leq x \leq 1.$$

Let $g(y)$ be the pdf of $Y = \frac{X}{1-X}$ So $x = \frac{y}{1+y}$

Now

$$\frac{dx}{dy} = (1-x)^2 = \left(1 - \frac{y}{1+y}\right)^2$$

Hence

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = \frac{1}{B(a,b)}\left(\frac{y}{1+y}\right)^{a-1}\left(1-\frac{y}{1+y}\right)^{b-1}\left(1-\frac{y}{1+y}\right)^{2}$$

$$= \frac{1}{B(a,b)}\frac{y^{a-1}}{(1+y)^{a+b}},\ 0 \le y \le \infty.$$

Which is the pdf of beta distribution of second kind. So *Y* follow a beta distribution of second kind.