Probability

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Probability

- Probability define a measure (i.e. to quantify) of the likelihood or the chances that an event E will occur. That is Probability provides mathematical models for random phenomena and experiments.
- Consider a random experiment and let E be an event in its sample S. let the experiment be repeated n times and let the event E happens f out of n repetitions. 'f' is called the frequency of E in 'n' repetition of the experiment and the ratio $\frac{f}{n}$ is called the frequency ratio or relative frequency of the event.
- The purpose of probability theory is to describe and predict such relative frequencies (averages) in terms of probabilities of events

Approaches to the Probability

There are three ways to assign probabilities to events:

- Classical approach,
- Relative-frequency approach
- Axiomatic approach
- Now consider each of them one by one

Classical or Mathematical or a Priori probability

If a random experiment or trials results in 'n' exhaustive, mutually exclusive and equally likely outcomes, out of which 'f' favorable to the occurrence of the event E, then the probability of occurrence of E is given by

$$P(E) = \frac{Number\ of\ favourable\ cases\ to\ E}{Total\ number\ of\ cases} = \frac{f}{n}$$

Example

1. In tossing a coin, what is the prob. of getting a head.

Sol: Total no. of events = {H, T}= 2
Favourable event = {H}= 1

$$\text{Probability} = \frac{Number\ of\ favourable\ cases}{Total\ number\ of\ exhaustive\ cases}$$

$$= \frac{1}{2}$$

2. In throwing a die, the prob. of getting 2.

Sol: Total no. of events = {1,2,3,4,5,6}= 6
Favourable event = {2}= 1

$$\text{Probability} = \frac{\textit{Number of favourable cases}}{\textit{Total number of exhaustive cases}}$$
$$= \frac{1}{6}$$

3. Find the prob. of throwing 7 with two dice.

Sol: Total no. of possible ways of throwing a dice twice = 36 ways
Number of ways of getting 7 is, (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) = 6

$$\text{Probability} = \frac{\textit{Number of favourable cases}}{\textit{Total number of exhaustive cases}}$$
$$= \frac{6}{36}$$
$$= \frac{1}{6}$$

Advantages

- The following are the advantages of the classical definition

1. It is simple and easily understandable even by a layman.
2. In cases where this definition is applicable the determination of the probability is easy.
3. To determine the probability of an event we require only the number of points in the sample space and the number of points in the event.
4. The probability of an event is obtained accurately and not as a limit which will be the case if the frequency definition is adopted
5. The filed of application of this definition is very wide.

Disadvantages or Limitation

1. This is not possible to apply when there are infinitely many outcomes in the sample space.
2. Nobody knows whether the outcomes are equally likely or not. In most cases it is not the case.
3. The definition finds application in a limited field only. Classical definition dealt with discrete variables.
4. Some complicated calculations may involve in some cases to find the number of points in the sample space and in the event.

Relative frequency approach

- Von Mises's Statistical Probability or Empirical Probability or relative frequency definition is as follows
- *If trails be repeated a greater number of times under, essentially the same conditions, the limit of the ratio of the number of times that an event happens to the total number of trials as the number of trials increases indefinitely is called the probability of that event.*
- Consider a random experiment and let *E* be an event in its sample *S*. let the experiment be repeated *n* times and let the event *E* happens *f* out of *n* repetitions. The probability that the event E will occur is

$$P(E) = \lim_{n \to \infty} \frac{f}{n}.$$

Advantages

1. It provides a practical method for estimating the probability of an event.
2. It is easily understood even by a layman.
3. No elaborate calculation is necessary to determine the probability of an event. That is the calculation involved for finding probabilities are not complicated.
4. It gives a clear idea about the concept of probability.
5. The probability of an event can be determined without making assumptions like equally likely
6. This definition increases the domain of application of probability theory in various branches of modern sciences and social sciences.

Disadvantages or Limitation

1. Only an estimate of the probability of an event can be obtained by this method
2. Even to get the estimate of the probability the random experiment has to be repeated a large number of times
3. In many practical situation it is difficult to repeat the experiment a large number of times under the same conditions in many practical situations.
4. If the experiment is hypothetical or imaginary this method can not be applied.

Axiomatic Probability

Axioms of Probability: Consider an experiment with the sample sample S. For each event E, we can associate a real number $P(E)$ such that:

Positivity :

(a) $P(E) \geq 0$,

Certainty :

(b) $P(S) = 1$,

Additivity :

(c) For each sequence of events $E_1, E_2, \ldots$ that are mutually exclusive (that is $E_i \cap E_j = \emptyset$ (the empty set), if $i \neq j$), we have

$$P\left(E_1 \bigcup E_2 \bigcup \ldots\right) = P(E_1) + P(E_2) + \ldots.$$

$P(E)$ is called the probability that E occurs.

Theorems of Probability

Theorem 1

For any event E, $0 \leq P(E) \leq 1$

Proof: Consider a random experiment and let E be an event in its sample S. let the experiment be repeated n times and let the event E happens f out of n repetitions. Then it is obvious, that f, is an integer such that $0 \leq f \leq n$.

Dividing through out with n we have $\frac{0}{n} \leq \frac{f}{n} \leq \frac{n}{n}$ which gives

$$0 \leq \frac{f}{n} \leq 1$$

Now taking limit we have $0 \leq \lim_{n \to \infty} \frac{f}{n} \leq 1$. Hence $0 \leq P(E) \leq 1$

Theorem 2

Let S be the sample space then $P(S) = 1$

Proof: By frequency definition $P(S) = \lim_{n \to \infty} \frac{f}{n}$, where f is the number of times the event S occurred out of the n repetitions of the random experiment. But in any repetition of the random experiment, the out come is an element of the sample space. Hence if the experiment is repeated for n times, the number of times S occurred is also n. Hence,

$$P(S) = \lim_{n \to \infty} \frac{n}{n} = 1.$$

Theorem 3

For any event E, $P(E)+P(E^c) = 1$ or $P(E) = 1 - P(E^c)$.

Proof: Consider a random experiment and let E be an event in its sample S. let the experiment be repeated n times and let the event E happens f out of n repetitions. Then obviously, E^c happens in n-f times. So by frequency definition

$$P(E^c) = \lim_{n \to \infty} \frac{n-f}{n} = \lim_{n \to \infty} \left(1 - \frac{f}{n}\right) = 1 - \lim_{n \to \infty} \frac{f}{n} = 1 - P(E)$$

or

$$P(E)+P(E^c) = 1.$$

Theorem 4

For the null event ϕ, $P(\phi) = 0$.

Proof: Let S be the sample space then $\phi = S^c$. So by theorem 2 $P(S) = 1$ and by theorem 3

$$P(\phi) = P(S^c) = 1 - P(S) = 1 - 1 = 0$$

Or

Out of n repetition, in no event ϕ occurs. That is for ϕ, f=0.

Hence $P(\phi) = \lim_{n \to \infty} \frac{0}{n} = 0$.

Theorem 5 (Addition Theorem)

For any two event A and B, $P(A \cup B) = P(A)+P(B)-P(A \cap B)$.

Proof: Let A and B be to events in a random experiment, which are not disjoint. Let the experiment be repeated n times. Let A occurs *f* times, B occurs *g* times and both A and B together happened for h times. Then total number of times A or B occurred is, *f* +*g*-*h*. Hence

$$P(A \cup B) = \lim_{n \to \infty} \left(\frac{f+g-h}{n}\right) = \lim_{n \to \infty} \frac{f}{n} + \lim_{n \to \infty} \frac{g}{n} - \lim_{n \to \infty} \frac{h}{n}.$$

$$= P(A)+P(B)-P(A \cap B).$$

Illustration

- Let the experiment be tossing a die. Then S={1,2,3,4,5,6} Let the event A be the number less than 5 turn up and B be the event that an even number turn up. Then A = {1,2,3,4} and B= {2,4,6}. Now $A \cup B$ ={1,2,3,4,6} and $A \cap B$ ={2,4}
- The number of elements in S, n=6, number of elements in A, f=4, n=6, number of elements in B, g=3. Now $f+g=7$ but the number of elements in $A \cup B$ is only 5 this because in both A and B, {2,4} occurs and when find union of A and B we count them only once. That is we have to subtract the number of elements in $A \cap B$, say h from $f+g$ to obtain the number of elements in $A \cup B$. In other words the number of elements in $A \cup B$ is $f+g-h$.

Note

- If A and B are mutually exclusive or disjoint then $A \cap B = \phi$ and
- $P(A \cap B) = P(\phi) = 0$. In this case $P(A \cup B) = P(A) + P(B)$

- Addition theorem for 3 variable
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$
- If $B \subset A$ then $P(B) \leq P(A)$.

Conditional Probability

- The probability of an event Aa, given that an event B of the same sample space has happened is called the conditional probability of A given B and is denoted as P(A | B).
- Definition
- Let A and B be two events. Then, the conditional probability of A given that B has occurred, P(A | B), is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0.$$

Similarly

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0.$$

Independence of events

- If the information that the event A has happened makes no change in the probability of the happening of the event B, we say that A and B are two independent events. If two event A and B can happened together we say that they are independent. Consider an experiment of tossing a coin and die together. Let A be the event that A shows head and B be the event that the die shows an even number. Here the occurrence of A does not prevent the occurrence of B. That is the coin shows head die show a even number can occur together. So A and B are independent.
- Definition

Two event A and B are said to be independent if

$$P(A \mid B) = P(A) \text{ or } P(B \mid A) = P(B) \text{ or } P(A \cap B) = P(A)P(B).$$

Note

- For two independent event A and B

$$P(A \mid B) = P(A),$$

and

$$P(B \mid A) = P(B).$$

and

$$P(A \cup B) = P(A) + P(B) - P(A)P(B).$$

Theorem 6 (Multiplication Theorem)

For any two event A and B, $P(A \cap B) = P(A)\, P(B|\, A)$, $P(A) \neq 0$.

Or $P(A \cap B) = P(B)\, P(A|\, B)$, $P(B) \neq 0$.

Proof: Let A and B be to events in a random experiment, which are not disjoint. Then by the definition of conditional probability

$$P(B|\, A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0.$$

Which by cross multiplication gives

$$P(A \cap B) = P(A)\, P(B|\, A).$$

The End

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