RANDOM VARIABLES

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RANDOM VARIABLE

- A real valued function defined over a sample space of a random experiment is called the random variable associated with the experiment. So a random variable is a real valued function from sample space to real line.
- For example, let us consider a coin tossing experiment. Here the sample space will be S = (H, T). Now define a variable X such that

$$X = \begin{cases} 1 & \text{if head turn up} \\ 0 & \text{other wise} \end{cases}$$

Then X is a Random Variable

TYPES OF RANDOM VARIABLES

- Random variable (r.v) can be either discrete or continuous.

Discrete Random variable:-

- A random variable is said to be discrete if its range includes finite number of values or countable infinite number of values. For example, Number of road accidents occurs in a day in a city.

Continuous random variable:-

- A random variable which can assume any value from a specified interval of the form [a,b] is known as continuous random variable . For example, Lifetime of mobile phone or height of a randomly selected student from your college etc can be treated as continuous r.vs.

PROBABILITY MASS FUNCTION

- Let R be the set of all possible values of a discrete random variable X; and $f(x) = P(X = x)$ for each x in R. Then $f(x)$ is called the probability mass function (pmf) or probability density function (pdf) of X. The expression $P(X = x)$ means the probability that X assumes the value x.
- **Properties**:-

1. As $f(x)$ is a probability function, $f(x) \geq 0$.
2. The total probability $\Sigma f(x) = 1$.

EXAMPLE - 1

Example 1.1 A fair coin is to be flipped three times. Let X denote the number of heads that can be observed out of these three flips. Then X is a discrete random variable with the set of possible values $\{0, 1, 2, 3\}$; this set is also called the *support* of X. The sample space for this example consists of all possible outcomes ($2^3 = 8$ outcomes) that could result out of three flips of a coin, and is given by

$$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Note that all the above outcomes are equally likely to occur with a chance of 1/8. Let A denote the event of observing two heads. The event A occurs if one of the outcomes HHT, HTH, and THH occurs. Therefore, $P(A) = 3/8$. The probability distribution of X can be obtained similarly and is given below:

k:	0	1	2	3
$P(X = k)$:	1/8	3/8	3/8	1/8

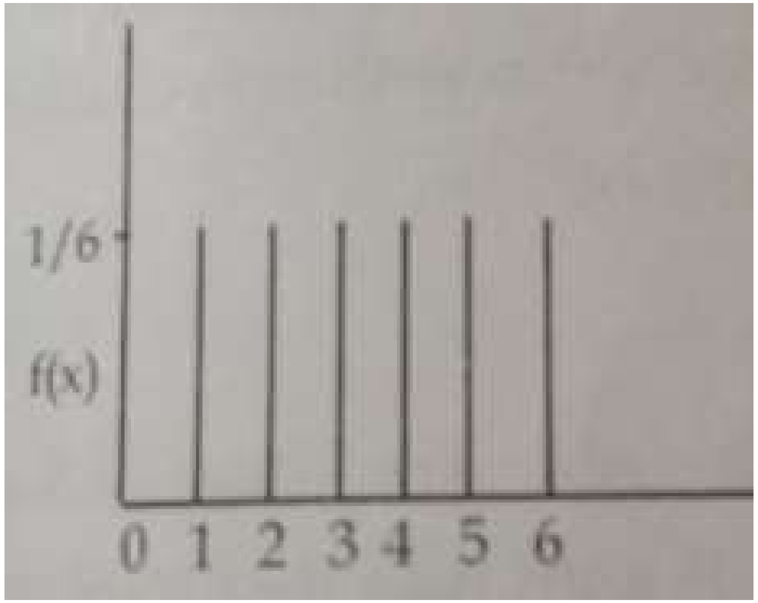
This probability distribution can also be obtained using the probability mass function. For this example, the pmf is given by

$$P(X = k) = \binom{3}{k} \left(\frac{1}{2}\right)^{k} \left(1 - \frac{1}{2}\right)^{3-k}, \quad k = 0, 1, 2, 3,$$

EXAMPLE - 2

- Consider an experiment of throwing an unbiased die. Let X denote the face turn up. Then X takes the values 1,2,3,4,5,6 and the probability Mass function is defined as

- $f(x) = P(X=x) = \frac{1}{6}, x = 1,2,3,4,5,6.$



DENSITY FUNCTION (DF)

- P(X≤x) is called the distribution function of X and is usually denoted by F(x). F(x) is also called cumulative density functions (cdf).

- **Properties of Distribution function**

If F(x) is the distribution function of a r.v. X, then

1. F(a) =0 where a is the value less than the lower limit of X. (F(-∞) =0).
2. F(b) =1 where b is the value more than the upper limit of X. (F(∞)=1).
3. If c < d, then F(c) ≤ F(d) (i.e F(x) is non decreasing).
4. F(x) is continuous from the right.
5. $f(x_i) = F(x_i) - F(x_{i-1})$.

DF for Example 1

x	f(x)	F(x)
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	1

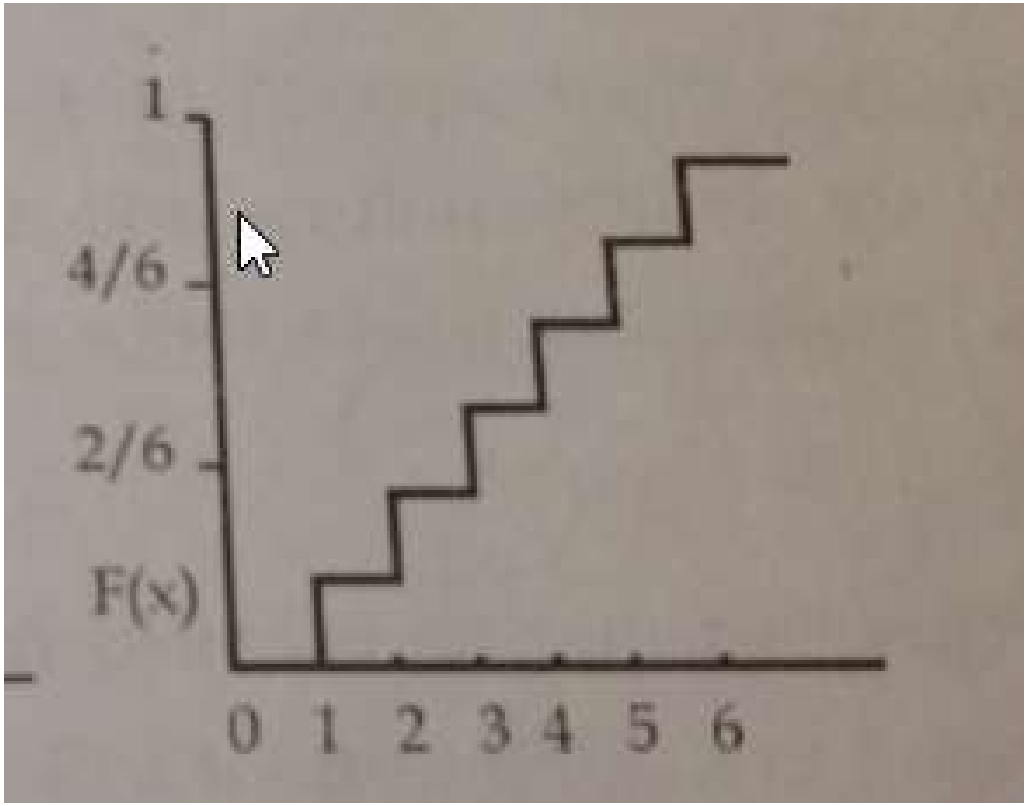
Hence the distribution function $F(x) = P(X \leq x)$ will be

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{8} & 0 \leq x < 1 \\ \frac{4}{8} & 1 \leq x < 2u \\ \frac{7}{8} & 2 \leq x < 3 \\ 1 & \geq 3 \end{cases}$$

DF for Example 2

x	f(x)	F(x)
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	1

Plot of F(x)



PROBABILITY DENSITY FUNCTION FOR CONTINUOUS RV

- Let X be a continuous random variable and F(x) = P(X ≤ x) be its distribution function of X. Let dx be a small positive quantity. Then the probability that the rv X lies in the interval [x, $x+dx$] is given as

$$P[x < X \leq x = dx] = F(x + dx) - F(x) = f(x)dx$$

- Here $f(x)$ is called the probability density function of X and $f(x)\ dx$ is called the probability element

MATHEMATICAL EXPECTATION

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the **expected value** of X is

$$E(X) = \sum_{x} x f(x).$$

Consider an urn containing n lots which has written $x_1, x_2, \dots, x_n$. Let a lot be drawn from the urn and let X denote the value on the lot. Then the pmf of X is given by

$f(x) = P(X=x) = \frac{1}{n}, x = x_1, x_2, \dots, x_n.$

So $E(X) = \sum_{i=1}^{n} x f(x) = \frac{x_1}{n} + \frac{x_2}{n} + \cdots + \frac{x_n}{n} = \frac{x_1+x_2+\cdots+x_n}{n} = \bar{x} =$ mean

EXAMPLE -3

- Let the pmf of the Random variable X is as follows find the Expected value of X.

x	0	1	2
$f(x)$	$6/11$	$9/22$	$1/22$

Now,

$$E(X) = \sum_{x=0}^{2} x f(x) = 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22} = \frac{1}{2}$$

PROPERTIES OF EXPECTATIONS

- **Result 1**

If a and b are any constants, then

$$E(aX + b) = aE(X) + b.$$

- Special Cases

If a is a constant, then

$$E(aX) = aE(X).$$

If b is a constant, then

$$E(b) = b.$$

- **Result 2**

If X *and* Y *are* **any** *random variables, then*

$$E(X+Y) = E(X) + E(Y).$$

- **Result 3**

If X *and* Y *are* **independent** *random variables, then*

$$E(X \cdot Y) = E(X) \cdot E(Y).$$

EXPECTATION OF A FUNCTION OF RANDOM VARIABLES

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the expected value of $g(X)$ is given by

$$E[g(X)] = \sum_{x} g(x)f(x).$$

EXAMPLE-4

If X is the number of points rolled with a balanced die, find the expected value of $g(X) = 2X^2 + 1$.

Solution. Since each possible outcome has the probability $\frac{1}{6}$, we get

$$
\begin{aligned}
E[g(X)] =& E[2X^2 + 1] = \sum_{x=1}^{6} g(x)f(x) = \sum_{x=1}^{6} (2x^2 + 1)\frac{1}{6} \\
=& (2 \cdot 1^2 + 1)\frac{1}{6} + (2 \cdot 2^2 + 1)\frac{1}{6} + \cdots + (2 \cdot 6^2 + 1)\frac{1}{6} \\
=& \frac{94}{3}.
\end{aligned}
$$

EXAMPLE-5

Suppose the probability distribution of the discrete random variable X is

x	0	1	2	3
$f(x)$	0.2	0.1	0.4	0.3

Find $E(X)$, $E(X^2)$. Knowing that, what is $E(4X^2)$ and $E(3X+2X^2)$?

Solution.

$E(X) = \sum_{x=0}^{3} xf(x) = 0 \cdot 0.2 + 1 \cdot 0.1 + 2 \cdot 0.4 + 3 \cdot 0.3 = 1.8,$

$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = 0^2 \cdot 0.2 + 1^2 \cdot 0.1 + 2^2 \cdot 0.4 + 3^2 \cdot 0.3 = 4.4,$

$E(4X^2) = 4E(X^2) = 4 \cdot 4.4 = 17.6,$

$E(3X+2X^2) = 3E(X) + 2E(X^2) = 3 \cdot 1.8 + 2 \cdot 4.4 = 14.2.$

MOMENTS

The rth **moment about the origin** of a random variable X, denoted by μ'_r, is the expected value of X^r; symbolically,

$$\mu'_r = E(X^r) = \sum_x x^r f(x)$$

for $r = 0, 1, 2, \cdots$ when X is discrete, and

When $r = 0$, we have $\mu'_0 = E(X^0) = E(1) = 1$.

When $r = 1$, we have $\mu'_1 = E(X^1) = E(X)$, which is just the expected value of the random variable X, and we give it a special symbol and a special name.

μ_1' is called the **mean** of the distribution function of X, or simply the **mean** of X, and it is denoted by μ.

The rth **moment about the mean** of a random variable X, denoted by μ_r, is the expected value of $(X-\mu)^r$; symbolically,

$$\mu_r = E[(X-\mu)^r] = \sum_x (x-\mu)^r f(x)$$

for $r = 0, 1, 2, \cdots$ when X is discrete, and

μ_2 is called the **variance** of the distribution of X, or simply the **variance** of X, and it is denoted by σ^2, $\text{var}(X)$, or $V(X)$; σ, the positive square root of the variance, is called the **standard deviation**.

PROPERTIES OF VARIANCE

- **Result-1**

If X has the variance σ^2, then

$$var(aX + b) = a^2\sigma^2.$$

- **Result-2**

If X and Y are independent random variables,

$$var(X + Y) = var(X) + var(Y), \; var(X - Y) = var(X) + var(Y).$$

EXAMPLE-6

Find μ, μ_2', and σ^2 for the random variable X that has the probability distribution $f(x) = \frac{1}{2}$ for $x = -2$ and $x = 2$.

Solution.

$$\mu = E(X) = \sum_x x f(x) = (-2)\frac{1}{2} + 2\frac{1}{2} = 0,$$

$$\mu_2' = E(X^2) = \sum_x x^2 f(x) = (-2)^2\frac{1}{2} + 2^2\frac{1}{2} = 4,$$

$$\sigma^2 = \mu_2 = \mu_2' - \mu^2 = 4 - 0^2 = 4,$$

or

$$\sigma^2 = \mu_2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x) = (-2-0)^2\frac{1}{2} + (2-0)^2\frac{1}{2} = 4.$$

The End

This is completes your first unit

Next we consider the Specific distributions

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