General LP Problem

Let $x \in E_n$ and $f(x)$ and $g(x)$ be linear function defined as

$$f(x) = \sum_{j=1}^{n} c_j x_j$$

and

$$g_i(x) = \sum_{j=1}^{n} a_{ij} x_j - b_i$$

$c_j, a_{ij}, b_j \in R, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n$

Then the LPP is defined as

Minimize $f(x)$

Subjected to the constrians

$$g_i(x) = 0 \qquad i = 1, 2, \ldots, m$$

and

$$x \geq 0.$$

Matrix Definition

Minimize $f(x) = cx$,

Subjected to

$$Ax = B, \quad x \geq 0$$

where C is a raw ~~matrix~~ vector and X and B are Column vectors ie $C = [c_1\ c_2 \cdots c_n]$ $X = [x_1\ x_2 \ldots x_n]'$

$B = [b_1\ b_2 \ldots, b_m]'$ and A is an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The equivalent form of the problem in ordinary notation is

$$\text{Minimize } f = \sum_{j=1}^{n} c_j x_j \quad \text{— (I)}$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, 2, \dots, m. \quad \text{— (II)}$$

and

$$x_j \geq 0 \quad j = 1, 2, \dots, n \quad \text{— (III)}$$

Equation (1) is the objective function to be minimized, (2) are the constrians and (3) the non-negativity condition. The coefficient c_j are usully called the cost coefficients.

Example

Write the following LPP. in the above standard form

Maximize $f = 2x_1 + x_2 - x_3$.

subjected to

$$2x_1 - 5x_2 + 3x_3 \leq 4$$

$$3x_1 + 6x_2 - x_3 \geq 2$$

$$x_1 + x_2 + x_3 = 4$$

$x_1 \geq 0$ $x_3 \geq 0$. x_2 is unrestricted

Here one variable x_2 is unrestricted which we want to convert into restricted variable. For this we replace the x_2 ~~two~~ with two variables x_{21}, x_{22} Such that $x_2 = x_{21} - x_{22}$, $x_{21} \geq 0$, $x_{22} \geq 0$ Next we want to convert the inequality into equality by introducing slack variables x_4 and x_5.

Minimize $\psi = (-f) = -2x_1 - x_{21} + x_{22} + x_3$

Subjected to

$$2x_1 - 5x_{21} + 5x_{22} + 3x_3 + x_4 = 4$$

$$3x_1 + 6x_{21} - 6x_{22} - x_3 - x_5 = 2$$

$$x_1 + x_{21} - x_{22} + x_3 = 4$$

$$x_1, x_{21}, x_{22}, x_3, x_4, x_5 \geq 0.$$

Feasible Solutions

A solution of (2) and (3) is called a feasible Solution.

Theorem-1

The set S_F of feasible solution, if not empty, is a closed convex set (polytope) bounded from below and so has at least one vertex

Proof

S_F is the intersection of the hyper planes $g_i(X) = 0$, $i = 1, 2, \dots, m$ and let $H\{X \mid X \geq 0\}$. All these are convex sets and H is bounded from below. Hence S_F is closed convex set (polytope) bounded from below. and so it has a vertex..

Alternative Proof

Let X_1 and X_2 be two feasible Solutions. Then $X_1 \geq 0$ and $X_2 \geq 0$ (1) and $AX_1 = B$, $AX_2 = B$ — (2)

Let X be any convex linear combination of X_1 and X_2. Then

$$X = (1-\lambda) X_1 + \lambda X_2 \quad 0 \leq \lambda \leq 1$$

Now from (1) we have

$$X \geq 0 \quad \text{———} \quad (3)$$

Further

$$AX = A\left[(1-\lambda)X_1 + \lambda X_2\right]$$

$$= (1-\lambda) AX_1 + \lambda AX_2 = B \quad \text{from} (2)$$

$$\text{———} \quad (4)$$

Equation (3) and (4) mean that X is a feasible Solution. Thus the convex linear combination of every two feasible solution is feasible solution. Therefore the set of feasible solution is Convex set.

Basic Solution

Now Consider the equation

$$Ax = B \quad \text{———} \quad (1)$$

which has m equations in n unknown. We assume that m < n and the equations are linearly independent

Now. equation (1) Can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{—} (2)$$

which can be written as

$$x_1 P_1 + x_2 P_2 + \dots + x_n P_n = B \quad \text{———} \quad (3)$$

where P_j, $j = 1, 2, \dots n$ is the m vector in the j th Column of A. Then it is possible to find $\alpha_j \in R$, $j = 1, 2, \dots m$ not all zeros such that $\alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_m P_m + \alpha_{m+r} P_{m+r} = 0$

where P_{m+r} is any of the remaining vectors of the set. Thus the vectors $P_{m+1}, P_{m+2}, \dots P_n$ can be ~~separ~~ separately expressed as the linear combination of $P_1, P_2, \dots, P_m$ and so (3) can be writen as

$$y_1 P_1 + y_2 P_2 + \dots + y_m P_m = B \quad \text{--- (4)}$$

Suppose that the m-vector $[\xi_1, \xi_2 \dots \xi_m]'$ is the solution of (4). Then the n-vector ~~$[\xi_1, \xi_2, \dots \xi_{10}]'$~~ $[\xi_1, \xi_2, \dots \xi_m, 0, 0 \dots 0]'$ is a solution of (3) and (1) so it is a basic solution of (1). Here the linear independent vectors $P_1, P_2, \dots P_m$ are a basis and the variables $x_1, x_2 \dots x_m$ are the basic variables

Basic Feassible Solutions

A basic solution of (II) Satisfying (III) is called basic feasible solution. (bfs).

Theorem 2

A basic feasible solution of the LP problem is a vertex of the convex set of feasible solutions, or equivalently, if a set of vectors $P_1\ P_2 \dots P_m$ can be found that are linearly independent such that

$$y_1 P_1 + y_2 P_2 + \dots + y_m P_m = B \quad \text{--- (1)}$$

and

$$y_j \geq 0, \quad j = 1, 2, \dots, m$$

Then $X_y = [y_1\ y_2 \dots y_m, 0, \dots, 0]'$

is a basic feasible solution and is an extreme point (vertex) of S_F

Proof

Since X_y is a bfs it belong to S_F

Now suppose it is not an extreme point

Then two points X_1 and X_2 differes from X_y exist in S_F such that for some $0<\lambda<1$

$$X_y = \lambda X_1 + (1-\lambda) X_2$$

that is $\quad y_j = \lambda x_{1j} + (1-\lambda) x_{2j} \quad j = 1, 2, \ldots, m$

and $\quad \lambda x_{1j} + (1-\lambda) x_{2j} = 0 \quad$ for $j = m+1, m+2, \ldots, n$

Since $X_1, X_2 \in S_F \quad x_{1j}, x_{2j} \geq 0$ and since $0<\lambda<1$

we have $\quad x_{1j} = x_{2j} = 0 \quad j = m+1, m+2, \ldots, n$

There fore $\quad X_1 = [x_{11}, x_{12} \ldots x_{1m}, 0, 0 \cdots 0]'$

and $\quad X_2 = [x_{21} \; x_{22} \cdots x_{2m}, 0, 0 \cdots, 0]'$

Since X_1 and X_2 are solution of $AX = B$ we have

$$x_{11} P_1 + x_{12} P_2 + \cdots + x_{1m} P_m = B \quad \text{——(2)}$$

and $$x_{21} P_1 + x_{22} P_2 + \cdots + x_{2m} P_m = B \quad \text{——(3)}$$

Using (1) and (2) we have

$$(y_1 - x_{11}) P_1 + (y_2 - x_{12}) P_2 + \cdots + (y_m - x_{1m}) P_m = 0$$

But $P_1 \, P_2 \cdots P_m$ are linearly independent by hypothesis

There fore $\quad y_1 = x_{11}, \; y_2 = x_{12}, \; \cdots, \; y_m = x_{1m}$ or

$$X_y = X_1.$$

which contradict the assumption that X_y is not an extreme point. Hence X_y is an extreme point

Theorem - 3 A vertex of S_F is a basic feasible solution

Proof

Let $X_y = [y_1\ y_2 \cdots y_n]'$ be a vertex of S_F. Then since $X_y \in S_F$, $X_y \geq 0$

Let r of the y_j's, $j = 1, 2, \ldots, n$ be non zero where $r \leq n$. Since $m < n$, either $r \leq m$ or $r > m$. If $r \leq m$, X_y is obviously a bfs so the theorem hold's.

Now let $r > m$, then we may put X_y as

$$X_y = [y_1\ y_2 \cdots y_r, 0, 0 \cdots 0]'$$

where $y_j > 0$ for $j = 1, 2, \ldots, r$.

Since X_y is a solution of $AX = B$. we have

$$y_1 P_1 + y_2 P_2 + \cdots + y_r P_r = B \quad \text{———} \quad (1)$$

Since $r > m$ here the vectors $P_1, P_2, \ldots, P_r$ are not linearly independent. Hence there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_r$ not all zero such that

$$\alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_r P_r = 0$$

Multiplying the above equation by $c > 0$ we get

$$c\alpha_1 P_1 + c\alpha_2 P_2 + \cdots + c\alpha_r P_r = 0 \quad \text{———} \quad (2)$$

Adding and subtrating (1) and (2) we get.

$$(y_1 + c\alpha_1)P_1 + (y_2 + c\alpha_2)P_2 + \cdots + (y_r + c\alpha_r)P_r = B \quad — (3)$$

and

$$(y_1 - c\alpha_1)P_1 + (y_2 - c\alpha_2)P_2 + \cdots + (y_r - c\alpha_r)P_r = B \quad — (4)$$

Choose $c > 0$ sufficiently small to make.

$$y_j \pm c\alpha_j > 0 \quad \text{for } j = 1, 2, \ldots, r$$

Then we can conclude from (3) and (4) That

$$X_1 = [\, y_1 + c\alpha_1,\ y_2 + c\alpha_2, \ldots, y_r + c\alpha_r,\ 0 \ldots 0 \,]'$$

and

$$X_2 = [\, y_1 - c\alpha_1,\ y_2 - c\alpha_2, \ldots, y_r - c\alpha_r,\ 0 \ldots 0 \,]'$$

are feasible solutions. Thus we have Three feasible Solutions X_y, X_1 and X_2 which are related as

$$X_y = \frac{1}{2} X_1 + \frac{1}{2} X_2 .$$

Hence X_y is a convex linear combination of X_1 and X_2 which are both different from X_y. This means that X_y is not a vertex which contradicts our initial assumption

Hence $r \geq m$, X_y is a basic feasible Solution.

<u>Optimal Solution</u> : A bfs of (II) and (III) which optimizes the objective function (I) is called an optimal solution of the LP Problem.

Simplex method

Because of the inequalities in the constrains it has not been posible to find an analytic solution to LP problem. So numerical methods are developed to find or compute the solution for numerical values of x_i's (for finite number). The most general and widely used of these methods is called simplex method

The simplex method provides an algorithm which consists in ~~move~~ moving from one vertex of S_F (one bfs) to another in a prescribed manner such that the value of the objective function $f(x)$ at the succeeding vertex is less than at the preceding vertex. The procedure of jumping from vertex to vertex is repeated. If we can reduce $f(x)$ at each jump, then no basis can ever repeat and we can never go back to a vertex already covered. Since the number of vertices is finite, the process must ~~to be~~ lead to the optimal vertex in a finite number of steps.

Canonical Form of Equations

Let $x_1, x_2, \ldots, x_m$ be the basic variables corresponding to a certain basis of the equations

$$AX = B \qquad \text{—— (1)}$$

This can then be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 - a_{1m+1}x_{m+1} - \cdots - a_{1n}x_n \\ b_2 - a_{2m+1}x_{m+1} - \cdots + a_{2n}x_n \\ \cdots \\ b_m - a_{mm+1}x_{m+1} + \cdots + a_{mn}x_n \end{bmatrix}$$

$$\text{—— (2)}$$

The $m \times m$ matrix on the RHS of (2) is nonsingular because the basic vectors which are the column of this matrix are linearly independent. Premultiplying both side by its inverse we get

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \bar{b}_1 - \bar{a}_{1\,m+1} x_{m+1} - \cdots - \bar{a}_{1n} x_n \\ \bar{b}_2 - \bar{a}_{2\,m+1} x_{m+1} - \cdots - \bar{a}_{2n} x_n \\ \cdots \\ \bar{b}_m - \bar{a}_{m\,m+1} x_{m+1} - \cdots - \bar{a}_{mn} x_n \end{bmatrix} \quad \text{— (3)}$$

or

$$\begin{aligned} x_1 + \bar{a}_{1\,m+1} x_{m+1} + \cdots + \bar{a}_{1n} x_n &= \bar{b}_1 \\ x_2 + \bar{a}_{2\,m+1} x_{m+1} + \cdots + \bar{a}_{2n} x_n &= \bar{b}_2 \\ \cdots \\ x_m + \bar{a}_{m\,m+1} x_{m+1} + \cdots + \bar{a}_{m\,m+1} x_n &= \bar{b}_m \end{aligned} \quad \text{— (4)}$$

Equation (4) which are equivalent to (1) are called the canonical form of the equations provided $\bar{b} \geq 0$ $i = 1, 2, \ldots, m$. Correspoding to each feasible basis we can get a canonical form and vice versa. The advantage of putting the equation in a cannonical form is that the basis and the corresponding b.f.s can be immediately found out. Since the non basic varcables has value zero by putting $x_{m+1} = x_{m+2} = x_{m+3} = \cdots = x_n = 0$ in (4) we get the bfs as $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m, 0, \ldots, 0)$ using (4) we can eliminate the basic variable from the objective functon can be writen as

$$f(x) = \sum_{i=1}^{m} \bar{b}_i c_i + \sum_{j=m+1}^{n} \bar{c}_j x_j \quad \text{——— (5)}$$

where $\bar{c}_j = c_j - \sum_{i=1}^{m} c_i \bar{a}_{ij}$ $j = m+1, \ldots n$

The advantage of (5) is that the value of $f(x)$ for the present bfs is immedietly obtained as $\sum_{i=1}^{n} \bar{b}_i c_i$. The coefficients $\bar{c}_j$ $j = m+1, \ldots, n$ are called the relative cost coefficient.

Example - 1

Let $x \in E_2$ and $f(x) = 4x_1 + 5x_2$ to be maximized subjet to

$$x_1 - 2x_2 \le 2$$
$$2x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 5$$
$$-x_1 + x_2 \le 2$$
$$\cancel{x_1 + x_2 \ge 1}$$
$$x_1\ x_2 \ge 0$$

Let $\psi(x) = -f(x)$ the restructured problem takes the form

$$\psi(x) = -4x_1 - 5x_2 \quad \text{———} \quad (1)$$

Subjected to

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 2x_1 + x_2 + x_4 &= 6 \\ x_3 + 2x_2 + x_5 &= 5 \\ -x_1 + x_2 + x_6 &= 2 \end{aligned} \qquad (2)$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

First canonical form

Take $x_1 = 0$, $x_2 = 0$ then $x_3 = 2$ $x_4 = 6$ $x_5 = 5$ and $x_6 = 2$ as bfs.
and the corresponding value of ψ is

$$\psi = -4x_1 - 5x_2 = 0 \quad \text{———} \quad (3)$$

which is not optimum.

Now let $x_1 = 0$ and we have to find the value x_2 such that all $x_i's \geq 0$. For this

put $x_3 = 0$ then $x_2 = -1$, $x_4 = 7$, $x_5 = 7$, $x_6 = 3$

$x_4 = 0$ then $x_2 = 6$, $x_3 = 14$, $x_5 = -7$, $x_6 = -4$

$x_5 = 0$ then $x_2 = 5/2$ $x_3 = 7$, $x_4 = 7/2$ $x_6 = 1/2$

$x_6 = 0$ then $x_2 = 2$ $x_3 = 6$ $x_4 = 4$ $x_5 = 1$.

Second Canonical form

It is easy to discover a simple rule for deciding which variable drop. Consider the ratio $2/(-2)$, $6/1$, $5/2$, $2/1$ of the right-hand side constant of each of the equations (2) to the coefficient of x_2 in that equation. Of the +ve one least is $2/1$ corresponding to the last equation which determines the maximum value which can be given to x_2 bringing it into the basis without facing any other variable to become negative

$$\begin{aligned}
-x_1 + 2x_6 + x_3 \quad &= 6 \\
3x_1 - x_6 \quad + x_4 \quad &= 4 \qquad (4) \\
3x_1 - 2x_6 \quad + x_5 &= 1 \\
-x_1 + x_6 \quad + x_2 &= 2
\end{aligned}$$

So second bfs is $x_3 = 6$, $x_4 = 4$, $x_5 = 1$ $x_2 = 2$ $x_1 = 0$ $x_6 = 0$

Also eliminate x_2 from (1)

$$-x_2 = -x_1 + x_6 - 2.$$

$$\psi(x) = -4x_1 + 5(-x_1 + x_6 - 2)$$

$$= -4x_1 - 5x_1 + 5x_6 - 10$$

$$\psi(x) + 10 = -9x_1 + 5x_6. \quad \text{———} \quad (5)$$

and the value of ~~bfs~~ ψ for present bfs is -10

Third canonical form

The new basic variables are x_1, x_2, x_3, x_4 and ~~x_5~~ the corresponding canonical form is

$$\frac{1}{3}x_5 + \frac{4}{5}x_6 + x_3 = \frac{19}{3}$$

$$-x_5 + x_6 + x_4 = 3. \qquad (6)$$

$$\frac{1}{3}x_5 - \frac{2}{3}x_6 + x_1 = \frac{1}{3}$$

$$\frac{1}{3}x_5 + \frac{1}{3}x_6 + x_2 = \frac{7}{3}$$

and the f expressed in terms of nonbasic variable is

$$f \mp 13 = 3x_5 - x_6 \qquad (7)$$

$$\cancel{= \frac{4}{3}x_4 + \frac{5}{3}x_5}$$

and the bfs is $x_1 = 1/3$ $x_2 = 7/3$ $x_3 = 19/3$ $x_4 = 3$ $x_5 = 0$ $x_6 = 0$

The cofficient of x_6 is (7) is negative so it is not optitimal

Fourth Canonical form

The new basic variables are x_1, x_2, x_3, x_6 and the Corresponding canonical form is

$$-\frac{4}{3}x_4 + \frac{5}{3}x_5 + x_3 = \frac{7}{3}$$

$$x_4 - x_5 + x_6 = 3.$$

$$\frac{2}{3}x_4 - \frac{1}{3}x_5 + x_1 = 7/3$$

$$-\frac{1}{3}x_4 + \frac{2}{3}x_5 + x_2 = 4/3$$

and the ψ expressed in terms of x_4, x_5 is

$$\psi + 16 = x_4 + 2x_5$$

here Coefficient x_4 and x_5 are positive So optimum value is attained. ~~and~~

~~So Maximum of $-f(x) + 16$~~

~~$f(x) =$~~

So Minimum of $\psi(x)$ is -16.

Hence the solution is $x_1 = 7/3$ $x_2 = 4/3$

$x_3 = 7/3$ $x_4 = 0$, $x_5 = 0$, $x_6 = 3$

and Maximum of $f(x) = 16$

Simplex Tableau

Iteration	Basis	B	P1	P2	P3	P4	P5	P6
I	x3	2	1	-2	1			
	x4	6	2	1		1		
	x5	5	1	2			1	
	x6	2	-1	1				1
	Ψ	0	-4	-5				
II	X3	6	-1		1			2
	X4	4	3			1		-1
	X5	1	3				1	-2
	X2	2	-1	1				1
	Ψ	10	-9					5
III	X3	19/3			1		1/3	4/3
	X4	3				1	-1	1
	X1	1/3	1				1/3	-2/3
	X2	7/3		1			1/3	1/3
	Ψ	13					3	-1
IV	X3	7/3			1	-4/3	5/3	
	X6	3				1	-1	1
	X1	7/3	1			-2/3	1/3	
	X2	4/3		1		-1/3	2/3	
	Ψ	16				1	2	

The first column in the table shows the iteration number. The second and the third column gives the variable in the basis and their values (vector **B**). The succeeding column gives the vectors $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots$ of the canonical form, which means that row wise the entries in these columns are the coefficient of $x_1, x_2, \ldots$ in the equations.

The following sequence of steps constitute one iteration leading from one b.f.s. to another. Let us start from I=1

1. Examine the relative cost coefficients. If all are non-negative, the current solution is optimal
2. If not, pick out the numerically largest negative coefficient (-5). The vector corresponding to it ($\mathbf{P}_2$) is to be brought into the basis. The corresponding basis variable is x_2.
3. Divide each element of vector **B** by the corresponding elements of the chosen column vector ($\mathbf{P}_2$). Out of the positive ratios choose the least (2/1). The corresponding basic variable (x_6) has to go out of the basis.
4. If all the ratios are negative, it means that the value of incoming variable (whatever it is), can be made as large as we please without violating the feasibility condition. It follows that the problem has an unbounded solution. Iteration stops.
5. Replace the x_6 by x_2 in the basic variables column in the table for the next iteration I=2 and rewrite the equation against it so that the coefficient of x_2 is 1. Eliminate x_2 from rest of the equation in such a way that the coefficient of the basic variables x_3, x_4, x_5 remains 1.
6. Eliminate x_2 from the equation for Ψ also so that it is expressed in terms of the new non basic variable x_1, x_6 only. The entry in the third column of the Ψ equation gives the value of -Ψ at this stage.
7. Thus the table for I=2 is complete Go to 1.

Example-2

Maximize $5x_1 + 3x_2$
Subjected to

$$4x_1 + 5x_2 \leq 10$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

$$\psi = f(-x) = -5x_1 - 3x_2 \quad \text{——— (1)}$$

s.t.

$$\left.\begin{aligned} 4x_1 + 5x_2 + x_3 \quad\quad\quad &= 10 \\ 5x_1 + 2x_2 \quad + x_4 \quad &= 10 \\ 3x_1 + 8x_2 \quad\quad + x_5 &= 12. \end{aligned}\right\} \quad (2)$$

First cannonical form

Take $x_1 = 0 \quad x_2 = 0$ then $x_3 = 10,\ x_4 = 10,\ x_5 = 12$ and ~~Now~~ (1) takes the value zero ie $\psi(x) = 0$. which is trivial solution

Now in (1) Coefficient of x_i is larger. So x_1 is the in coming variable. From equation (2) we have $\frac{b_i}{a_{i1}}$ ratio for x_1 is $\frac{10}{4}$, $10/2$, $12/2$ ie 2.5, 2, 6. So the second equation of (2) gives the minimum value so $x_4 = 0$ is the new zero variables. ~~Ato~~.

Second Iteration

Removing x_2 from first and last equation of (2) and after making the coefficient of x_1 in second equation to (1)

$$\left.\begin{aligned} \frac{17}{5}x_2 - \quad + x_3 \quad - \frac{4}{5}x_4 &= 2 \\ x_1 + \frac{2}{5}x_2 - \quad\quad + \frac{1}{5}x_4 &= 2 \\ \frac{34}{5}x_2 \cancel{-\tfrac{3}{5}} \quad\quad - \frac{3}{5}x_4 + x_5 &= 6 \end{aligned}\right\} \quad (3)$$

are the new constrians $\psi(x)$ can be fund out from second equation $-x_1 = -2 + \frac{2}{5}x_2 + \frac{1}{5}x_3$

$$\psi(x) = -5x_1 - 3x_2$$
$$= -10 + 2x_2 + x_4 - 3x_2$$
$$\psi(x) + 10 = -x_2 + x_4 \quad -(4)$$

This has -ve coefficient so the optimality does not attain

Third Iteration

From 4 x_2 is the new bfs. Now the ratio $\frac{b_i}{a_{2i}}$ gives the value $\frac{10}{17}$, 5, $\frac{30}{34}$ which gives the minimum value $\frac{10}{17}$. So we keep $x_3 = 0$ of the first equation of (3)

$$\left.\begin{aligned} x_2 + \frac{5}{17}x_3 - \frac{4}{17}x_4 &= 10/17 \\ x_1 \quad + \frac{2}{17}x_3 + \frac{9}{85}x_4 &= 30/17 \\ -2x_3 + x_4 + x_5 &= 2 \end{aligned}\right\} \quad (5)$$

From (5) $-x_2 = -\frac{10}{17} + \frac{5}{17}x_3 - \frac{4}{17}x_4$

So $\psi(x) + 10 = -\frac{10}{17} + \frac{5}{17}x_3 - \frac{4}{17}x_4 + x_4$

$$\psi(x) + \frac{180}{17} = \frac{5}{17}x_3 + \frac{13}{17}x_4$$

So optimality is attained and optimum values are for $x_1 = \frac{30}{17}$ $x_2 = \frac{10}{17}$, $x_3 = 0$, $x_4 = 0$, $x_5 = 2$

with $\psi(x) = \frac{-180}{17}$

& Max $f(x) = \frac{180}{17}$.

Simplex table

Iteration	Basis	B	P_1	P_2	P_3	P_4	P_5
I	x_3	10	4	5	1		
	x_4	10	5	2		1	
	x_5	12	3	8			1
ψ		0	-5	-3			
II	x_3	2	0	$17/5$	1	$-4/5$	
	x_1	2	1	$2/5$		$1/5$	
	x_5	6	0	$34/5$		$-3/5$	1
ψ		10	-1			$+1$	
III	x_2	$10/17$		1	$\frac{5}{17}$	$\frac{-4}{17}$	
	x_1	$30/17$	1		$\frac{2}{17}$	$\frac{9}{85}$	
	x_5	2			-2	1	1
ψ		$\frac{180}{17}$			$5/17$	$13/17$	

Minimize $f(x) = 4x_1 + 5x_2$

subject to

$$2x_1 + x_2 \le 6$$

$$x_1 + 2x_2 \le 5$$

$$x_1 + x_2 \geq 1$$

$$x_1 + 4x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Introducing the slack variable

$$\begin{aligned}
2x_1 + x_2 + x_3 \quad\quad\quad &= 6 \\
x_1 + 2x_2 \quad + x_4 \quad\quad &= 5 \\
x_1 + x_2 \quad\quad - x_5 \quad &= 1 \\
x_1 + 4x_2 \quad\quad\quad - x_6 &= 2
\end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

The initial solution $x_1 = 0$ $x_2 = 0$ $x_3 = 6$ $x_4 = 5$ $x_5 = -1$ and $x_6 = -2$ is not a bsf. To solve this problem we formulate a new auxiliary problem by introducing artificial variable x_7 and x_8 as follows

$$\text{Minimize} \quad g(x) = x_7 + x_8 \quad \text{——— (1)}$$

Subjected to

$$\begin{aligned}
2x_1 + x_2 + x_3 \quad\quad\quad &= 6 \quad \text{—— (2a)} \\
x_1 + 2x_2 \quad + x_4 \quad\quad &= 5 \quad \text{—— (2b)} \\
x_1 + x_2 \quad - x_5 + x_7 &= 1 \quad \text{—— (2c)} \\
x_1 + 4x_2 \quad - x_6 + x_8 &= 2 \quad \text{—— (2d)}
\end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

The solution of this problem may be $g(x) = 0$ with $x_7 = x_8 = 0$ and the values of the other variables are non negative with at least two of them are zero. But the optimal solution should be a bfs with at most 4 variables having non zero values. Then the values of the variables other than the artificial ones should constitute a bfs of the problem which can become the starting point of iteration of that problem. Thus we have 2 phases in this case. In first phase we proceed with problem II and in phase II with problem one after optimizing problem(s) eleminating the artificial variables

Now consider Phase I

from (2c) we have

$$x_7 = 1 - x_1 - x_2 + x_5$$

and from (2d)

$$x_8 = 2 - x_1 - 4x_2 + x_6$$

subtituting (1) we get

$$g(x) - 3 = -2x_1 - 5x_2 + x_5 + x_6 \quad \text{---} (3)$$

Iteration II

In (3) the coefficient of x_2 has maximum absolute value. to x_2 has to be enter in bfs. The ratio criteria gives the b/a_{i2} ratios as $6, 5/2, 1, 1/2$ So x_6 has to leave the bfs. Hence new contrians are

$$\frac{7}{4}x_1 + x_3 + \frac{1}{4}x_6 - \frac{1}{4}x_8 = 11/2 \quad \text{---(4a)}$$

$$1/2\, x_1 + x_4 + 1/2\, x_6 - 1/2\, x_8 = 4 \quad \text{---(4b)}$$

$$3/4\, x_1 - x_5 + \frac{1}{4}x_6 + x_7 - 1/4\, x_8 = 1/2 \quad (4c) \text{---} (1)$$

$$1/4\, x_1 + x_2 - 1/4\, x_6 + 1/4\, x_8 = 1/2 \quad (4d)$$

From (4d). $\Rightarrow -x_2 = -1/2 + 1/4\, x_1 - 1/4\, x_6 + 1/4\, x_8$

which gives $g(x) - 1/2 = \frac{-3}{4}x_1 + x_5 - 1/4\, x_6 + 5/4\, x_8$ —(5)
with $x_1 = 0 \quad x_2 = 1/2 \quad x_3 = 11/2 \quad x_4 = 4 \quad x_5 = -1/2 \quad x_6 = 0, x_7 = 0, x_8 = 0$
and $f(x) = 5/2$

Iteration III

Since $g(x)$ contain negative coefficient optimality is not achived. Since among the −ve coefficients the coefficient of x_1 is maximum. so in the next iteration x_1 has to be entered in bfs and the ratio $\frac{b_i}{a_{i1}}$ in (4) gives values $\frac{22}{7}, 8, 2/3, 2$ which indicate that 3rd equation gives the minimum ratio so x_5 has to be replaced with x_1 which gives

$$x_3 + \frac{7}{3}x_5 - 1/3\, x_6 - 7/3\, x_7 + \frac{1}{3}x_8 = 13/3 \quad \text{—(6a)}$$

$$x_4 + 2/3\, x_5 + \frac{1}{3}x_6 - 2/3\, x_7 \mp 1/3\, x_8 = 11/3 \quad \text{—(6b)}$$

$$x_1 \quad -4/3\, x_5 + \frac{1}{3}x_6 + 3/4\, x_7 \;\cancel{+}\; -1/3\, x_8 = 2/3 \quad \text{—(6c)}$$

$$x_2 \quad + \frac{1}{3}x_5 - 1/3\, x_6 - 1/3\, x_7 + 1/3\, x_8 = 1/3 \quad \text{—(6d)}$$

which gives

$$-x_1 = -1/2 - 3/5\, x_5 + 1/4\, x_6 + x_7 - 1/4\, x_8$$

Subtituting in (5) we get $g(x) = 2/5\, x_5 + x_7 + x_8$ —(7)

Since all the terms in RHS of (7) are positive optimality is attained. In this case we have $x_1 = 2/3$, $x_2 = 1/3$, $x_3 = 13/3$ and $x_4 = 11/3$.

Phase II

Rewriting the equations (6) in order we have.

~~$x_1 - 3/4\, x_5$~~

$$x_1 + 0x_2 + 0x_3 + 0x_4 - 4/3\, x_5 + \frac{1}{3}x_6 + 3/4\, x_7 + -1/3\, x_8 = 2/3$$

$$x_2 + 0x_3 + 0x_4 + 1/3\, x_5 - 1/3\, x_6 - 1/3\, x_7 + 1/3\, x_8 = 1/3$$

$$x_3 + 0x_4 + 7/3\, x_5 - 1/3\, x_6 - 7/3\, x_7 + 1/3\, x_8 = 13/3$$

$$+ x_4 + 2/3\, x_5 + 1/3\, x_6 - 2/3\, x_7 + -1/3\, x_8 = 11/3$$

$f(x) = (4 \times 2/3) + (5 \times 1/3) = 13/3$

Simplex table

Phase I	Bases	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
I	x_3	6	2	1	1					
	x_4	5	1	2		1				
	x_7	1	1	1			-1		1	
	x_8	2	1	4				-1		1
$g(x)$		-3	-2	-5			1	1		
f		0	4	5			0	0		
II	x_3	11/2	7/4		1			1/4		-1/4
	x_4	4	1/2			1		1/2		-1/2
	x_5	1/2	3/4				-1	1/4	1	-1/4
	x_2	1/2	1/4	1				-1/4		1/4
$g(x)$		-1/2	-3/4				1	-1/4		5/4
$f(x)$		-5/2	11/4					5/4		-5/4
III	x_3	13/3			1		7/3	-1/3	-7/3	1/3
	x_4	11/3				1	2/3	1/3	-2/3	-1/3
	x_1	2/3	1				-4/3	1/3	4/3	1/3
	x_2	1/3		1			1/3	-1/3	-1/3	1/3
$g(x)$		0					2/5		1	1
$f(x)$		-13/3								

Phase II

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
x_1	2/3	1				-4/3	1/3
x_2	1/3		1			1/3	-1/3
x_3	13/3			1		7/3	-1/3
x_4	11/3				1	2/3	1/3
$f(x)$	-13/3					11/3	1/3

Ans.

$\text{Min } f(x) = \frac{13}{3}$

$x_1 = 2/3$

$x_2 = 1/3$.

In general we define the auxillary (or phase I) for the LPP minimization problem as

$$\text{Minimize} \quad g(x) = \sum_{i=1}^{m} x_{n+i}$$

Subjected to

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_j \,, \quad i = 1, 2, \ldots, m$$

$$X \geq 0$$

where $X = [x_1, x_2, \ldots x_n, x_{n+1}, \ldots, x_{n+m}]$; Here x_{n+i} being called the artificial variables.

In this problem $\min g(x) = 0$ if and only if $x_n = 0$ for all i. Hence if we solve this problem by the simplex method we get its solution as $g(x) = 0$ only if in its optimal b.f.s the ~~article~~ artificial variables are zero. The optimal values of the rest of the variables, being non-negative will then satisfy the constriants of the original problem. Moreover, not more than m of these variables being non zero, they will constitute a basic feasible solution of the original problem providing a starting point for its solution by the simplex method.

If $\min g(x) > 0$, the conclusion is that there is no feasible solution of auxilliary ~~variable~~ problem with the value of the artificial variables as zero, and consequently no feasible solution exist for the original LP problem

An alternative method to solve the problems using Big M method. In this method the original objective function f is replaced by $F = f + M \sum_{i=1}^{m} x_{n+i}$, where M is an arbitrary large numbers as compaired to the coefficent (C_i) in f and x_{n+i} are artificial variables.

Degeneracy

The least of a set of non-negative ratios decides which variables is to be dripped from the basis at a particular stage. It is to be dropped from the basis at a particular stage. It may happen that two or more ratios are equal and that the least. In that case a tie occurs as to decide which variable to drop. We can arbitrarily decide in favour of one, but then it turns out that other variables which tied with it and continue to remain in the basis also become zero. That is we have one or more of the basic variables too have zero value. Such a case is called degeneracy.

Simplex Multipliers

Consider the LPP

$$\text{Minimize } f(X) = CX \qquad (1)$$

Subjected to

$$AX=B \qquad (2)$$

$$X\geq 0 \qquad (3)$$

Suppose $(x_1, x_2, \dots, x_m, 0, \dots, 0)$ is a b.f.s. To express f(X) in terms of the non-basic variables $x_{m+1}, \dots, x_n$, we may eliminate the basic variable $x_1, x_2, \dots, x_m$, from (1) with the help of (2). For this let us multiply a each of the equations of (2) by constants $\pi_1, \pi_2, \dots, \pi_m$ respectively and add them to (1), which gives

Minimize $\quad f(X)= \sum_{j=i}^{n} c_j x_j \qquad (1)$

Subjected to

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \qquad i=1,2,\dots,m \qquad (2)$$

$$x_j \geq 0, \qquad j=1,2,\dots,n \qquad (3)$$

$$\left(c_1 + \sum_{i=1}^{m} a_{i1}\pi_i\right)x_1 + .. + \left(c_n + \sum_{i=1}^{m} a_{in}\pi_i\right)x_n = f + \sum_{i=1}^{m} \pi_i b_i$$

or

$$\sum_{j=1}^{n}\left(c_j + \sum_{i=1}^{m} a_{ij}\pi_i\right)x_j = f + \sum_{i=1}^{m} \pi_i b_i \qquad (4)$$

Now chose $\pi_1, \pi_2, \dots, \pi_m$ such that

$$\sum_{i=1}^{m} a_{ij}\pi_i = -c_j, j=1,2,\dots, m \qquad (5)$$

Then (4) reduces to

$$f = \sum_{j=m+1}^{n} \bar{c}_j - \sum_{i=1}^{m} \pi_i b_i \qquad (6)$$

where

$$\bar{c}_j = c_j + \sum_{i=1}^{m} a_{ij}\pi_i \qquad (7)$$

(5) are *m* equations *m* in unknowns π_i and this constants are called *simplex multipliers*.

Now in Matrix notation we put (5) as

$$A_0'\Pi = -C_0'$$

where

$$A_0 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdot\cdot & a_{mn} \end{bmatrix}, \Pi = [\pi_1, \pi_2, \dots, \pi_m]' \text{ and } C_0 = [c_1, c_2, \dots, c_n]$$

So we have

$$\Pi = -[A_0']^{-1}C_0' = -[A_0^{-1}]'C_0' = -C_0A_0^{-1}. \qquad (8)$$

The vector Π is called the multiplier vector and it component are called *simplex multipliers*.

Example:

Consider the example which we done in the case of Artificial variables.

Minimize f(X) = $4x_1 + 5x_2$

Subjected to

$$2x_1 + x_2 \le 6$$

$$x_1 + 2x_2 \le 5$$

$$x_1 + x_2 \ge 1$$

$$x_1 + 4x_2 \ge 2$$

$$x_1, x_2 \ge 0$$

The initial and final tables of the above problem is

Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_3	6	2	1	1			
x_4	5	1	2		1		
x_5	1	1	1			-1	
x_6	2	1	4				-1
f	0	4	5				
x_1	2/3	1				-4/3	1/3
x_2	1/3		1			1/3	-1/3
x_3	13/3			1		7/3	-1/3
x4	11/3				1	2/3	1/3
f	-13/3					11/3	1/3

Note that x_1, x_2, x_3, x_4, are the basic variables in the optimal solution. To obtain Π we have to find out A_0^{-1} where A_0 is the matrix of the coefficient of these variables in the initial form i.e.

$$A_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{bmatrix}$$

A_0^{-1} operating in the initial matrix of the coefficients produces the final coefficient matrix that is

$$A_0^{-1} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 4 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -4/3 & 1/3 \\ 0 & 1 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 7/3 & -1/3 \\ 0 & 0 & 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

Taking only the submatrix of the last four column on either side

$$A_0^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 \\ 1 & 0 & 7/3 & -1/3 \\ 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

Since the inverse of a diagonal matrix with entries 1 or -1 is the matrix it self we get

$$A_0^{-1} = \begin{bmatrix} 0 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 \\ 1 & 0 & 7/3 & -1/3 \\ 0 & 1 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix}$$

Now the simplex multiplayer using (8) can be obtained as

$$[\pi_1 \; \pi_2 \;\; \pi_3 \;\; \pi_4] = -[4\;5\;0\;0] \begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix}$$

$$= \left[0\;0 - \tfrac{11}{3} - \tfrac{1}{3}\right].$$

Revised Simplex Method

Revised Simplex Method which is a modification of the original simplex method which is more economical. The economy in the revised simplex method is not working out the entire simplex tableau but calculating only the numbers required in the following essential steps

1. For the feasible basis x_j, j=1,2, …, m, the coefficient $\bar{c}_j$, j= m +1, m+2, …, n, of the equation $f = \sum_{i=1}^{m} \bar{b}_i c_i + \sum_{j=m+1}^{n} \bar{c}_j$ are directly calculated. For this, let A_0 be the matrix of a feasible basis. Then compute the multiplayer vector is computed using $\Pi = -C_0 A_0^{-1}$. Then $\bar{c}_j = c_j + \sum_{i=1}^{m} a_{ij}\pi_i$

2. Let one of these coefficients, say $\bar{c}_r$ be negative. So we choose to x_r, be brought into the basis. The numbers $\bar{a}_{ir}$ and $\bar{b}_i$ *i*=1,2, …, *m*, in the column for x_r, in the equation

$$x_1 + \bar{a}_{1m+1}x_{m+1} + \cdots + \bar{a}_{1n}x_n = \bar{b}_1$$
$$x_2 + \bar{a}_{2m+1}x_{m+1} + \cdots + \bar{a}_{2n}x_n = \bar{b}_2$$
$$\ldots.$$
$$x_m + \bar{a}_{mm+1}x_{m+1} + \cdots + \bar{a}_{mn}x_n = \bar{b}_m$$

are now directly computed using

$$\begin{bmatrix} \bar{a}_{1r} \\ \bar{a}_{2r} \\ \cdot \\ \vdots \\ \bar{a}_{mr} \end{bmatrix} = A_0^{-1} \begin{bmatrix} a_{1r} \\ a_{2r} \\ \cdot \\ \vdots \\ a_{mr} \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \cdot \\ \vdots \\ \bar{b}_m \end{bmatrix} = A_0^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \vdots \\ b_m \end{bmatrix}$$

3. Let

$$\frac{\bar{b}_p}{\bar{a}_{pr}} = \min \frac{\bar{b}_i}{\bar{a}_{ir}}, \ \bar{a}_{ir} > 0.$$

So we replace by to get the new basis. The matrix for the new basis is obtained directly from the initial matrix *A*.

The sequence of the operations (1), (2) and (3) are repeated till we get the optimal solution.

Note: Let

$$A_{0k}^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{23} \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{m1} & \beta_{m2} & \ldots & \beta_{mm} \end{bmatrix}$$

be the inverse matrix in the *k* the iteration then the inverse matrix in the next iteration

$$A_{0k+1}^{-1} = \begin{bmatrix} \beta_{11}^* & \beta_{12}^* & \ldots & \beta_{1m}^* \\ \beta_{21}^* & \beta_{22}^* & \ldots & \beta_{23}^* \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{m1}^* & \beta_{m2}^* & \ldots & \beta_{mm}^* \end{bmatrix}$$

can be obtained using the formula

$\beta_{pj}^* = \frac{\beta_{rj}}{\bar{a}_{pr}}$, *j*=1,2,…,m and

$\beta_{ij}^{*} = \beta_{ij} - \bar{a}_{ir}\beta_{pj}^{*}$, i=1,2,…,m, j=1,2,…,m, $i \neq p$

Example

Minimize $f(x) = x_1 + x_2 + x_3$

Subjected to

$$
\begin{aligned}
x_1 \qquad\qquad\qquad & -x_4 \quad -2x_6 = 5 \\
x_2 \qquad\qquad & +2x_4 - 3x_5 + x_6 = 3 \\
x_3 \qquad & +2x_4 - 5x_5 + 6x_6 = 5
\end{aligned}
$$

$$x_j \geq 0, j=1,2,\ldots,6$$

Let us rewrite the inequalities above

Minimize $f(x) = x_1 + x_2 + x_3 + 0x_4 + 0x_5 + 0x_6$

$$x_1 + 0x_2 + 0x_3 - x_4 + 0x_5 - 2x_6 = 5$$

$$0x_1 + x_2 + 0x_3 + 2x_4 - 3x_5 + x_6 = 3$$

$$0x_1 + 0x_2 + x_3 + 2x_4 - 5x_5 + 6x_6 = 5$$

Here

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & 2 & -5 & 6 \end{bmatrix}$$

$$B = [5 \quad 3 \quad 5]', C=[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]$$

Iteration 1. A b.f.s which provide starting point is obviously

$$x_1 = 5, x_2 = 3, x_3 = 5, \qquad x_4 = x_5 = x_6 = 0$$

So we have

$$A_{01} = A_{01}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_{01} = [1 \quad 1 \quad 1]$$

Using (8) we get

$$[\pi_1 \, \pi_2 \;\, \pi_3\,] = -C_{01}A_{01}^{-1} = [-1 \quad -1 \quad -1]$$

From (7) we have

$$\bar{c}_j = c_j + \sum_{i=1}^{m} a_{ij}\pi_i$$

So

$$\bar{c}_4 = c_4 + \textstyle\sum_{i=1}^{3} a_{i4}\pi_i = 0+(-1\times-1)+(2\times-1)+(2\times-1)$$

$$= 0+1-2-2=-3$$

$$\bar{c}_5 = c_5 + \sum_{i=1}^{3} a_{i5}\pi_i = 0+(0\times-1)+(-3\times-1)+(-5\times-1)$$
$$= 0+0+3+5=8$$
$$\bar{c}_6 = c_6 + \sum_{i=1}^{3} a_{i6}\pi_i = 0+(-2\times-1)+(1\times-1)+(6\times-1)$$
$$= 0+2-1-6=-5$$

Since $\bar{c}_6$ is negative and maximum value we choose r=6 as is a unit matrix, we have

$$\begin{bmatrix}\bar{a}_{16}\\ \bar{a}_{26}\\ \bar{a}_{36}\end{bmatrix} = A_{01}^{-1}\begin{bmatrix}a_{16}\\ a_{26}\\ a_{36}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 1\\ 6\end{bmatrix}$$

$$[\bar{a}_{16} \quad \bar{a}_{26} \quad \bar{a}_{36}] = [a_{16} \quad a_{26} \quad a_{36}] = [-2 \quad 1 \quad 6]$$

and

$$\begin{bmatrix}\bar{b}_1\\ \bar{b}_2\\ \bar{b}_3\end{bmatrix} = A_{01}^{-1}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\ 3\\ 5\end{bmatrix}$$

$$[\bar{b}_1 \quad \bar{b}_2 \quad \bar{b}_3] = [b_1 \quad b_2 \quad b_3] = [5 \quad 3 \quad 5]$$

Now in absolute value

min $\frac{\bar{b}_i}{\bar{a}_{i6}} = 5/6 = \frac{\bar{b}_3}{\bar{a}_{36}}$ so we choose p=3.

$$A = \begin{bmatrix}1 & 0 & 0 & -1 & 0 & -2\\ 0 & 1 & 0 & 2 & -3 & 1\\ 0 & 0 & 1 & 2 & -5 & 6\end{bmatrix}$$
$$B = [5 \quad 3 \quad 5]', \; C=[1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]$$

Iteration 2. Now new basic variables are x_1, x_2, x_6 with new

$$A_{02} = \begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 1\\ 0 & 0 & 6\end{bmatrix}, C_0 = [1 \quad 1 \quad 0]$$

Now we wanted to find A_{02}^{-1} one way is to find the inverse directly. Second method is to find out the inverse using the formula

$\beta_{pj}^* = \frac{\beta_{rj}}{\bar{a}_{pr}}$, $\quad$ j=1,2,…,m

and

$\beta_{ij}^* = \beta_{ij} - \bar{a}_{ir}\beta_{pj}^*$, $\quad$ i=1,2,…,m, j=1,2,…,m, $i \neq p$.

Where β_{ij} are elements of A_{01}^{-1} and β_{ij}^* are the elements of A_{02}^{-1}

Here p=3 and m = 3 So we first compute

$$A_{01}^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\beta_{3j}^* = \frac{\beta_{3j}}{\bar{a}_{36}}, \qquad j=1,2,3$$

$\beta_{31}^* = \frac{\beta_{31}}{\bar{a}_{36}} = \frac{0}{6} = 0$, $\beta_{32}^* = \frac{\beta_{32}}{\bar{a}_{36}} = \frac{0}{6} = 0$, $\beta_{33}^* = \frac{\beta_{33}}{\bar{a}_{36}} = \frac{1}{6}$,

Now we compute the other elements of A_0^{-1}

$\beta_{ij}^* = \beta_{ij} - \bar{a}_{ir}\beta_{pj}^*$, $\qquad$ i=1,2,3, j=1,2,3, $i \neq p$.

$\beta_{11}^* = \beta_{11} - \bar{a}_{16}\beta_{31}^*$ = 1- (-2×0) = 1

$\beta_{12}^* = \beta_{12} - \bar{a}_{16}\beta_{32}^*$ = 0- (-2×0) = 0

$\beta_{13}^* = \beta_{13} - \bar{a}_{16}\beta_{33}^*$ = 0- $\left(-2 \times \frac{1}{6}\right) = \frac{2}{6}$

Similarly we get $\beta_{21}^* = 0$, β_{22}^*=1 and β_{32}^*=$\frac{-1}{6}$

So

$$A_{02}^{-1} = \begin{bmatrix} \beta_{11}^* & \beta_{12}^* & \beta_{13}^* \\ \beta_{21}^* & \beta_{22}^* & \beta_{23}^* \\ \beta_{31}^* & \beta_{32}^* & \beta_{33}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2/6 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1/6 \end{bmatrix}$$

$$C_0 = [1 \quad 1 \quad 0]$$

from this we get $[\pi_1 \; \pi_2 \;\; \pi_3 \;] = -C_0A_{02}^{-1} = [-1 \quad -1 \quad -1/6]$

and $\bar{c}_3 = \frac{5}{6}$, $\bar{c}_4 = \frac{-8}{6}$ and $\bar{c}_5 = \frac{23}{6}$. $\bar{c}_4$ is negative so r =4.

$$\begin{bmatrix} \bar{a}_{14} \\ \bar{a}_{24} \\ \bar{a}_{34} \end{bmatrix} = A_{02}^{-1} \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2/6 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 5/3 \\ 1/3 \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} = A_{02}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2/6 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 40/6 \\ 13/6 \\ 5/3 \end{bmatrix}$$

Now in absolute value

min $\frac{\bar{b}_i}{\bar{a}_{i4}}$ = 13/10 = $\frac{\bar{b}_2}{\bar{a}_{24}}$ so we choose p=2.

Iteration 3. Now new basic variables are x_1, x_4, x_6 with new

$$A_{03} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 2 & 6 \end{bmatrix}, C_0 = [1 \quad 0 \quad 0]$$

$$\begin{bmatrix}\bar{a}_{14}\\ \bar{a}_{24}\\ \bar{a}_{34}\end{bmatrix} = A_{02}^{-1}\begin{bmatrix}a_{14}\\ a_{24}\\ a_{34}\end{bmatrix} = \begin{bmatrix}1 & 0 & 2/6\\ 0 & 1 & -1/6\\ 0 & 0 & 1/6\end{bmatrix}\begin{bmatrix}-1\\ 2\\ 2\end{bmatrix} = \begin{bmatrix}-1/3\\ 5/3\\ 1/3\end{bmatrix}$$

With r=4 and p=2, the new A_{03}^{-1} is

$A_{03}^{-1} = \begin{bmatrix}1 & 1/5 & 9/30\\ 0 & 3/5 & -3/30\\ 0 & -1/5 & 6/30\end{bmatrix}$ and $[\pi_1\ \pi_2\ \ \pi_3\ ] = [-1 \quad -1/5 \quad -3/10]$.

Which gives $\bar{c}_2 = \frac{1}{2}$, $\bar{c}_3 = \frac{7}{10}$ and $\bar{c}_5 = \frac{21}{10}$.

All $\bar{c}_j$'s are positive so optimum is achieved. There for present basis is optimal. The value of the basic variables are given by

$$\begin{bmatrix}\bar{b}_1\\ \bar{b}_2\\ \bar{b}_3\end{bmatrix} = \begin{bmatrix}1 & 1/5 & 9/30\\ 0 & 3/5 & -3/30\\ 0 & -1/5 & 6/30\end{bmatrix}\begin{bmatrix}5\\ 3\\ 5\end{bmatrix} = \begin{bmatrix}71/10\\ 13/10\\ 2/5\end{bmatrix}$$

So we have the optimum solution as

$$x_1 = \frac{71}{10}, x_2 = 0, x_3 = 0, \qquad x_4 = \frac{13}{10}, x_5 = 0, x_6 = \frac{2}{5}$$

and optimum f(X)=$\frac{71}{10}$.

Table

I	Basis	C_0	A_0^{-1}			Π	$\bar{c}_j$	r	$\bar{a}_{ir}$	$\bar{b}_i$	p
	x_1	1	1	0	0	-1	$\bar{c}_4$ =-3		-2	5	
1	x_2	1	0	1	0	-1	$\bar{c}_5$ =8	6	1	3	3
	x_3	1	0	0	1	-1	$\bar{c}_6$ =-5		6	5	
	x_1	1	1	0	2/6	-1	$\bar{c}_3 = 5/6$		-1/3	40/6	
2	x_2	1	0	1	-1/6	-1	$\bar{c}_4 = -8/6$	4	5/3	13/6	2
	x_6	0	0	0	1/6	-1/6	$\bar{c}_5 = 23/6$		1/3	5/6	
	x_1	1	1	0	0	-1	$\bar{c}_2 = 1/2$	Optimum		71/10	
3	x_4	0	0	1	0	-1	$\bar{c}_3 = 7/10$	solution		13/10	
	x_3	0	0	0	1	-1	$\bar{c}_5 = 21/10$	reached		2/5	

Dual - Primal problems

Associated with any LPP is another LPP called the ***dual***. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LPP, the given LPP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min problem and vice versa.

The general linear programming problem has the form

Primal	Dual
Max Z = $c_1x_1 + c_2x_2 + .. + c_nx_n$. Subject to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$. $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$ … $a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n \le b_m$. $x_1, x_2, \ldots, x_n \ge 0$.	Min W = $b_1y_1 + b_2y_2 + .. + b_my_m$. Subject to $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1$ $a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2$ … $a_{1n}y_1 + a_{2n}y_2 + .. + a_{nm}y_m \ge c_n$. $y_1, y_2, \ldots, y_m \ge 0$.

Comparing the two problem we have the following points

1. If the primal contains n variables and m constraints, the dual will contains m variables and n constraints.
2. The maximization problem in the primal become the minimization problem in the dual and vice versa.
3. The maximization problem has (≤) constraints while the minimization problem has (≥) constraints.
4. The constants $c_1, c_2, \ldots, c_n$ in the objective function of the primal appear in the constraints of the duals.
5. The constants $b_1, b_2, \ldots, b_n$ in the constraints of the primal appear in the objective function of the duals.
6. The variables in both problem are non negative.

   The constraint relationship of the primal and dual can be represented in a single table as shown below

	x_1	x_2	x_3	…	x_n	
y_1	a_{11}	a_{12}	a_{13}	…	a_{1n}	$\le b_1$
y_2	a_{21}	a_{22}	a_{23}	…	a_{2n}	$\le b_2$
y_3	a_{31}	a_{32}	a_{33}	…	a_{3n}	$\le b_3$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
y_m	a_{m1}	a_{m2}	a_{m3}	…	a_{mn}	$\le b_m$
	$\ge c_1$	$\ge c_2$	$\ge c_3$	…	$\ge c_n$	

Example-1

Primal	Dual
Maximize Z= $3x_1+5x_2$ Subject to $2x_1+6x_2 \le 50$ $3x_1+2x_2 \le 35$ $5x_1-3x_2 \le 10$. $x_1, x_2, \ge 0$.	Minimize W= $50y_1+35y_2+10y_3$ Subject to $2y_1+3y_2+5y_3 \ge 3$ $6y_1+2y_2-3y_3 \ge 5$ $y_1, y_2, y_3, \ge 0$.

Example-2

Primal		Dual
Max Z= $3x_1+5x_2$ Subject to $2x_1+6x_2 \le 50$ $3x_1+2x_2 \le 35$ $5x_1-3x_2 \le 10.$ $x_1 \ge 2$ $5x_1+6x_2=20$ $x_1, x_2, \ge 0.$	Max Z= $3x_1+5x_2$ Subject to $2x_1+6x_2 \le 50$ $3x_1+2x_2 \le 35$ $5x_1-3x_2 \le 10.$ $-x_1+0\,x_2 \le -2$ $5x_1+6x_2 \le 20$ $-5x_1-6x_2 \le -20$ $x_1, x_2, \ge 0.$.	Min W=$50y_1+35y_2+10y_3-2y_4+20y_5-20y_6$ Subject to $2y_1+3y_2+5y_3-y_4+5y_5-5y_6 \ge 3$ $6y_1+2y_2-3y_3+0y_4+6y_5-6y_6 \ge 5$ $y_j \ge 0.$

Example-3

Primal	Dual	Dual of Dual
Min Z= $3x_1+5x_2$ Subject to $2x_1+6x_2 \ge 36$ $3x_1+2x_2 \ge 24$ $x_1, x_2, \ge 0.$	Max W= $36y_1+24\,y_2$ Subject to $2y_1+3y_2 \le 3$ $6y_1+2y_2 \le 5$ $y_1, y_2 \ge 0.$	Min Q= $3u_1+5u_2$ Subject to $2u_1+6u_2 \ge 36$ $3u_1+2u_2 \ge 24$ $u_1, u_2, \ge 0.$

Duality General Definition

We have the general LP Problem as

Minimize $f(\mathbf{X}) = CX$ (1)

Subjected to $AX \ge B$ (2)

$X \ge 0$ (3)

Where A is an mxn matrix. X and B are column n vector and X is a raw n vector. Then it's Duel is written as

Maximize $\phi(\mathbf{Y}) = B'Y$ (4)

Subjected to $A'Y \ge C'$ (5)

$Y \ge 0$ (6)

Where Y is column m vector.

Duality Theorems

Thoerem-1 The dual of the dual is the primal

Proof: Consider the duel (4), (5) and (6) this can be written as

Minimize $-\phi(\mathbf{X}) = -B'Y$

Subjected to $-A'Y \ge -C'$, $Y \ge 0$

Its duel according to the definition is

Maximize $\Psi(\mathbf{X}) = -CX$
Subjected to $-AX \leq -B$, $X \geq 0$

This may be rewrite as

Minimize $f(\mathbf{X}) = CX$
Subjected to $AX \geq B$,
$X \geq 0$

Which is same as the primal (1), (2) (3)

Thoerem-2 (Weak Duality Theorem)**:** The value of the objective function f(X) for any feasible solution of the primal is not less than the value of the objective function $\phi(\mathbf{Y})$ for any feasible solution of the dual. (ie. To prove min f(X) $\geq$ max $\phi(\mathbf{Y})$)

Proof: Consider the primal and dual problems

Primal	Dual
Min $f(X) = c_1x_1 + c_2x_2 + .. + c_nx_n$. Subject to $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$. $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2$ $\ldots$ $a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n \geq b_m$. $x_1, x_2, \ldots, x_n \geq 0$.	Max $\phi(\mathbf{Y}) = b_1y_1 + b_2y_2 + .. + b_my_m$. Subject to $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \leq c_1$ $a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \leq c_2$ $\ldots$ $a_{1n}y_1 + a_{2n}y_2 + .. + a_{nm}y_m \leq c_n$. $y_1, y_2, \ldots, y_m \geq 0$.

Now let us introduce the necessary slack variables to the above equations we have

Primal: Min $f(X) = c_1x_1 + c_2x_2 + .. + c_nx_n$.

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - x_{n+1} = b_1. \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n - x_{n+2} = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n - x_{n+m} = b_m. \\ \\ x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \geq 0. \end{array}\right. \qquad \mathbf{(1)}$$

Dual: Max $\phi(\mathbf{Y}) = b_1y_1 + b_2y_2 + .. + b_my_m$.

Subject to

$$\left.\begin{array}{l} a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m + y_{m+1} = c_1 \\ a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m + y_{m+2} = c_2 \\ \ldots \\ a_{1n}y_1 + a_{2n}y_2 + .. + a_{nm}y_m + y_{m+n} = c_n. \\ \\ y_1, y_2, \ldots, y_m, y_{m+1}, \ldots, y_{m+n} \geq 0. \end{array}\right. \qquad \mathbf{(2)}$$

Now let $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots., x_{n+m}$ be any feasible solution of the primal and $y_1, y_2, \ldots, y_m, y_{m+1}, \ldots, y_{m+n}$ be any feasible solution of the dual. Now multiplying the equation of constrains (1) with $y_1, y_2, \ldots, y_m$, and (2) with and $x_1, x_2, \ldots, x_n$, and finding the difference second - first we get

$$(c_1x_1 + c_2x_2 + .. + c_nx) - (b_1y_1 + b_2y_2 + .. + b_my_m)$$
$$= x_1\, y_{m+1} + x_2\, y_{m+2} + \ldots + x_n\, y_{m+n} + y_1\, x_{n+1} + y_2\, x_{n+2} + \ldots + y_m\, x_{n+m}.$$

i.e.

$$f-\phi = x_1\, y_{m+1} + x_2\, y_{m+2} + \ldots + x_n\, y_{m+n} + y_1\, x_{n+1} + y_2\, x_{n+2} + \ldots + y_m\, x_{n+m}. \qquad (3)$$

Since all the variable in the RHS of the (3) are nonnegative (as they are feasible solution) we have their sum and products are nonnegative. So

$$f - \phi \geq 0. \qquad (4)$$

Corollary-1: It immediately follow from equation (4) that

$$\min f(X) \geq \max \phi(\mathbf{Y})$$

Explanation

Consider the three variable problem

Primal	Dual
Min $f(X) = c_1x_1 + c_2x_2 + c_3x_3$. Subject to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - x_4 = b_1$. $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - x_5 = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - x_6 = b_3$ $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.	Max $\phi(\mathbf{Y}) = b_1y_1 + b_2y_2 + b_3y_3$. Subject to $a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + y_4 = c_1$ $a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + y_5 = c_2$ $a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + y_6 = c_3$ $y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$.

Now multiplying the equation of constrains primal with y_1, y_2, y_3 and dual with and $x_1, x_2, x_3,$ and finding the difference second - first we get

$$\begin{aligned} a_{11}x_1\, y_1 + a_{12}x_2\, y_1 + a_{13}x_3\, y_1 - x_4\, y_1 &= b_1\, y_1. \\ a_{21}x_1\, y_2 + a_{22}x_2\, y_2 + a_{23}x_3\, y_2 - x_5\, y_2 &= b_2\, y_2 \\ a_{31}x_1\, y_3 + a_{32}x_2\, y_3 + a_{33}x_3\, y_3 - x_6\, y_3 &= b_3\, y_3 \end{aligned} \qquad \text{(I)}$$

$$\begin{aligned} a_{11}\, x_1y_1 + a_{21}\, x_1y_2 + a_{31}\, x_1y_3 + x_1\, y_4 &= c_1\, x_1 \\ a_{12}\, x_2y_1 + a_{22}\, x_2y_2 + a_{32}\, x_2y_3 + x_2y_5 &= c_2\, x_2 \\ a_{13}\, x_3y_1 + a_{23}\, x_3y_2 + a_{33}\, x_3y_3 + x_3y_6 &= c_3\, x_3 \end{aligned} \qquad \text{(II)}$$

(II)-(I) gives .

$$x_1\, y_4 + x_2y_5 + x_3y_6 + x_4\, y_1 + x_5\, y_2 + x_6\, y_3 = (c_1\, x_1 + c_2\, x_2 + c_3\, x_3) - (b_1\, y_1 + b_2\, y_2 + b_3\, y_3)$$
$$= f(X) - \phi(Y)$$

Thoerem-3 (Optimality Criterion Theorem**):** The optimum value of the primal f(X) if exist, is equal to the optimum value of the dual $\phi(\mathbf{Y})$.

Proof:

Consider the primal f(X) [problem 1]

Minimize $\quad f(X)= \sum_{j=i}^{n} c_j x_j \quad (1)$

Subjected to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \qquad i=1,2,\ldots,m \qquad (2)$$

$$x_j \geq 0, \qquad j=1,2,\ldots,n \qquad (3)$$

Then its dual $\phi(\mathbf{Y})$ is [problem 2]

Maximize $\quad \phi(\mathbf{Y}) = \sum_{i=i}^{m} b_i y_i \quad (4)$

Subjected to

$$\sum_{i=1}^{m} a_{ij} y_i \leq c_j, \qquad j=1,2,\ldots,n \qquad (5)$$

$$y_i \geq 0, \qquad i=1,2,\ldots,m \qquad (6)$$

Introduce m surplus variables in (2) [problem 1] we get,

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i, \qquad i=1,2,\ldots,m \qquad (7)$$

Let the primal has optimal solution $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_n)$.
Since this has to be basic feasible solution, at least m of these numbers are zero. Let $\pi_1, \pi_2, \ldots, \pi_m$ be the simplex multiplayers for this solution, Then we have

$$f(X)+\sum_{j=i}^{m} \pi_i b_i = \sum_{j=1}^{n}\left(c_j + \sum_{i=1}^{m} a_{ij}\pi_i\right) x_j - \sum_{i=1}^{m} \pi_i x_{n+i} \qquad (8)$$

Since f(X) is optimum we have

$$\text{Min } f(X) = -\sum_{j=i}^{m} \pi_i b_i$$

and all the relative cost coefficients are non-negative, that is

$$c_j + \sum_{i=1}^{m} a_{ij}\pi_i \geq 0, j=1,2,\ldots,n$$

$$\pi_i \geq 0, \qquad i=1,2,\ldots,m$$

Or

$$-\sum_{i=1}^{m} a_{ij}\pi_i \leq c_j, \qquad -\pi_i \geq 0$$

The last 2 inequalities mean that $(-\pi_1, -\pi_2, \ldots, -\pi_m)$ is a solution of (6) and (7) that is it is a feasible solution of the dual. So we have

$$\phi(\mathbf{Y}) = -\sum_{j=i}^{m} \pi_i b_i = \text{Min } f(X) \qquad (9)$$

Thus we have a feasible solution of dual such that

Now by Corollary-1 this is possible only if

$$\text{Min } f(X) = \max \phi(\mathbf{Y})$$

Hence the solution of the dual is optimal.

Theorem 4: If the primal problem is feasible, then it has an unbounded optimum if and only if the dual has no feasible solution and vice versa.

Assume that the primal [problem 1] have an unbounded optimum. Then f(X) has no lower bound.

We know that Min f(X) $\geq$ Max ϕ(Y) (by Corollary-1)

$\Rightarrow$ ϕ(Y) has no solution (because, if there exist a solution it will be the lower bound of the function f(X)). So, if the primal is unbounded then the dual is infeasible.

Now assume that the primal is feasible but not unbounded. Then there exists a solution for the primal. Then from theorem 3, Min f(X) = Max ϕ(Y) and hence the dual is not infeasible.
$\Rightarrow$ if the dual is infeasible then the primal is unbounded.
Similarly, we can prove that dual is unbounded if and only if primal is infeasible.

Theorem 5 (Main Duality Theorem): If both the primal and the dual problems are feasible, then they both have optimal solutions such that their optimal values of the objective functions are equal.

From theorem 4, if both the primal and dual are feasible then both have optimum solutions.

From theorem 3, if the optimum value of the primal exist, it is equal to the optimum value of the dual.

(If ask to prove please prove theorem 3 and 4)

Theorem 6 (Complementary Slackness Theorem**):** If in the optimum solutions of the primal and dual,

1. a primal variable is positive, then the corresponding dual slack (surplus) variable is zero.
2. If a primal slack (surplus) variable is positive, then the corresponding dual variable is zero and vice versa.

Proof:

It follows from theorems 2 and 3 for the optimal solutions x_j, j=1,2,…,n,n+1, …n+m. of the primal and y_i, i=1,2,…,m, m+1,…, m+n of the dual should satisfy

$$x_1 y_{m+1} + x_2 y_{m+2} + \ldots + x_n y_{m+n} + y_1 x_{n+1} + y_2 x_{n+2} + \ldots + y_m x_{n+m} = 0$$

Since the optimal solution is feasible for all $x_j \geq 0$ and $y_i \geq 0$.

Hence all the terms in the expression on the left side above are non-negative and their sum are zero. So each term separately should be zero. So it is easy to verify that if $x_j > 0$ then $y_{m+j} = 0$, since $x_j y_{m+j} = 0$. Similarly if $y_i > 0$ then $x_{n+i} = 0$, since $y_i y_{n+i} = 0$.

Note: This theorem is helpful in determining the optimal solution of the primal from the optimum solution of the dual or vice versa.

Possibilities for		Primal Solution		
		Infeasible	Unbounded	Optimum
Dual Solution	Infeasible	Yes	Yes	No
	Unbounded	Yes	No	No
	Optimum	No	No	Yes

The following is an example how we can determine the optimal solution of primal using the optimal solution of dual

Consider the problem

Maximise $f = 3x_1 + 2x_2 + x_3 + 4x_4$,
Subject to

$$2x_1 + 2x_2 + x_3 + 3x_4 \le 20,$$
$$3x_1 + x_2 + x_3 + 2x_4 \le 20,$$
$$x_1, x_2, x_3, x_4 \ge 0,$$

Its dual is
Minimize $\phi = 20y_1 + 20y_2$
Subject to

$$2y_1 + 3y_2 \ge 3,$$
$$2y_1 + y_2 \ge 2,$$
$$y_1 + 2y_2 \ge 1,$$
$$3y_1 + 2y_2 \ge 4,$$
$$y_1, y_2 \ge 0.$$

This is a two variable problem whose solution can be obtained geometrically as

$$y_1 = 1.2,\ y_2 = 0.2,\ \phi = 28$$

After introducing the slack variables, the primal and dual constraints are

$$2x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 20,$$
$$3x_1 + x_2 + 2x_3 + 2x_4 + x_6 = 20,$$
$$2y_1 + 3y_2 - y_3 = 3,$$
$$2y_1 + y_2 - y_4 = 2,$$
$$y_1 + 2y_2 - y_5 = 1,$$
$$3y_1 + 2y_2 - y_6 = 4,$$
$$x_1, x_2, \ldots x_6,\ \ y_1, y_2, \ldots y_6 \ge 0,$$

Substituting the optimal values of $y_1 = 1.2$ and $y_2 = 0.2$ in the dual constraints, it follows that the slack variables

$y_3 = y_6 = 0$, $y_4 > 0$, $y_5 > 0$.
Thus the second and third constraints are satisfied as strict inequalities, and so the corresponding primal variables should be zero, that is, $x_2 = 0$, $x_3 = 0$.

Also since the dual variables, $y_1 > 0$, $y_2 > 0$, it follows that the corresponding primal constraints should be zero, that is, $x_5 = x_6 = 0$. The primal constraints thus reduce to

$2x_1 + 3x_4 = 20$
$3x_1 + 2x_4 = 20$

Which give $x_1 = 4$, $x_4 = 4$.

Thus optimal solution of the primal is therefore

$x_1 = x_4 = 4$, $x_2 = x_3 = 0$, $f = 28$

Application of Duality

We can find the optimal solution of the primal from the optimum solution of the dual problem. This co-existence of the solutions of the primal-dual pair is helpful in many practical situations. The following are the instances of the application of this result.

1. Usually in a l.p.p numerical work increases more with the number of constraints than the number of variables. So, if the number of constraints in primal problem is considerably larger than number of variables in it, then we can solve the dual problem with a smaller number of constraints. This approach is more economical than solving the primal problem.

2. In some situations, the dual problem can eliminate the use of artificial variables and hence the two-phase method. If the phase 1 of the two-phase method fails to eliminate the artificial variable from the basis, then we cannot move to the phase 2 and in such situation, we cannot find the optimum solution. In such situations we can use the dual problem to find the optimum solution.

3. Using the primal-dual relationship we can define a modified simplex method called dual simplex method, in which we start the iterations with an infeasible basic solution of the primal under certain conditions and proceed the iterations which will leads to the optimum solution of the primal.

This method is widely used when we have to enter a new constraint after solving the given problem. So, this method is more economical since it avoids the solution of the problems from the very beginning.

Dual Simplex Method

Dual simplex method is a highly modified form of simplex algorithm. We use this method when the ordinary simplex method fails with an initial feasible solution and do not want to proceed with artificial variables and hence the usage of two-phase methods. So, the usage this method is limited to the lpp with constraints in the form of $\geq$ inequality.

Dual simplex method utilises the relations between the properties of primal – dual pair. Consider a lpp with n variables and m constraints in which some of them are $\geq$ inequalities. Let the lpp (in standard form) be,

$$\text{Min. } f(\mathbf{X}) = \mathbf{C}'\mathbf{X} \qquad \text{subject to} \qquad \mathbf{AX} \geq \mathbf{b} \quad \text{and} \quad \mathbf{X} \geq \mathbf{0}.$$

Now the dual of the problem is,

$$\text{Max. } g(\mathbf{Y}) = \mathbf{b}'\mathbf{Y} \qquad \text{subject to} \qquad \mathbf{A}'\mathbf{Y} \leq \mathbf{C} \quad \text{and} \quad \mathbf{Y} \geq \mathbf{0}.$$

Now Introduce m surplus variables $\mathbf{S_1}$ in [problem 1] and n slack variables $\mathbf{S_2}$ in [problem 2] we get,

$$\text{Min. } f(\mathbf{X}) = \mathbf{C}'\mathbf{X} \qquad \text{subject to} \qquad \mathbf{AX} - \mathbf{S_1} = \mathbf{b} \quad \text{and} \quad \mathbf{X}, \mathbf{S_1} \geq \mathbf{0}.$$

$$\text{Max. } g(\mathbf{Y}) = \mathbf{b}'\mathbf{Y} \qquad \text{subject to} \qquad \mathbf{A}'\mathbf{Y} + \mathbf{S_2} = \mathbf{C} \quad \text{and} \quad \mathbf{Y}, \mathbf{S_2} \geq \mathbf{0}.$$

If $\mathbf{b} \leq \mathbf{0}$ and $\mathbf{C} \geq \mathbf{0}$ then, $\mathbf{S_1} = -\mathbf{b}$ and $\mathbf{S_2} = \mathbf{C}$ are basic feasible solutions of problem 1 and problem 2 respectively. Since $\mathbf{S_1}$ is a surplus vector its cost values are 0 and hence the value of the objective function of problem 1, at basic feasible solution $\mathbf{S_1} = -\mathbf{b}$, is 0**.** 0 is the least possible value of $f(\mathbf{X})$ and hence problem 1 is optimum. Then the optimum value of the dual is also 0 [Min. $f(\mathbf{X})$ = Max. $g(\mathbf{Y}) = 0$]. Hence $\mathbf{S_2} = \mathbf{C}$ is the optimum basic feasible solution of the dual.

However, if $\mathbf{b} \nleq \mathbf{0}$ (some b_i values positive) and $\mathbf{C} \geq \mathbf{0}$ then $\mathbf{S_1} = -\mathbf{b}$ is a basic infeasible solution of the primal. But $\mathbf{S_2} = \mathbf{C}$ is basic feasible solution of the dual. This condition is called **primal infeasible and dual feasible**. Then we start to find the optimum solution of the dual through a succession of basic feasible solutions of the dual ($\bar{\mathbf{C}} \geq \mathbf{0}$) till the relative cost values are non-positive ($\bar{\mathbf{b}} \leq \mathbf{0}$). Then find *the negative of the simplex multipliers for the dual and these are the values of the variables for the optimal solution of the primal*.

An algorithm is designed in accordance with the above theory and it is called **dual simplex algorithm** and it is given below.

Dual Simplex Algorithm

Consider the lpp (in standard form),

$$\text{Min. } f(\mathbf{X}) = \mathbf{C}'\mathbf{X} \quad \text{subject to} \quad \mathbf{AX} \geq \mathbf{b} \quad \text{and} \quad \mathbf{X} \geq \mathbf{0}.$$

That is,
$$\text{Min. } f(\mathbf{X}) = \mathbf{C}'\mathbf{X} \quad \text{subject to} \quad -\mathbf{AX} \leq -\mathbf{b} \quad \text{and} \quad \mathbf{X} \geq \mathbf{0}.$$

Now Introduce m surplus variables $\mathbf{S_1}$ in the constraints we get,

$$\text{Min. } f(\mathbf{X}) = \mathbf{C}'\mathbf{X} \quad \text{subject to} \quad -\mathbf{AX} + \mathbf{S_1} = -\mathbf{b} \quad \text{and} \quad \mathbf{X}, \mathbf{S_1} \geq \mathbf{0}.$$

If $\mathbf{b} \leq \mathbf{0}$ and $\mathbf{C} \geq \mathbf{0}$ then, $\mathbf{S_1} = -\mathbf{b}$ is the optimum solution of the problem. Otherwise if $\mathbf{b} \nleq \mathbf{0}$ (some b_i values positive) and $\mathbf{C} \ngeq \mathbf{0}$ then both the primal and dual basis are infeasible, and we cannot proceed the iterations to find the optimum solution. So, the method is a failure.

So, if $\mathbf{b} \nleq \mathbf{0}$ (some b_i values positive) and $\mathbf{C} \geq \mathbf{0}$ then proceed as follows.

1. Find an initial infeasible solution to the given lpp.

   The solution is infeasible implies that at least one of the basic variables has a negative value. $\mathbf{S_1} = -\mathbf{b}$ is the initial infeasible solution of the primal. Then the relative cost vector is $\mathbf{C} \geq \mathbf{0}$. In the dual simplex method, we cannot proceed to next step if at least one of the relative costs is negative.
2. Convert the current infeasible basic solution to a feasible one. For this, choose the basic variable which has the least value (largest negative value) and remove it from the basis. Let it be s_p.
3. Now we must convert a non-basic variable to a basic variable inorder to keep the strength of the basis unchanged (as m).

   The row corresponding to s_p is,

   $$-a_{p1}x_1 - a_{p2}x_2 - a_{p3}x_3 - \dots - a_{pr}x_r - \dots - a_{pn}x_n + s_p = -b_p.$$

   If $a_{pj} < 0$ for all $j = 1, 2, \dots n$ then we cannot make the right side constant positive and hence the problem has no feasible solution. Otherwise ($a_{pj} > 0$ for at least one j) we can proceed the iterations to find the optimum solution.

   Assume that $a_{pr} > 0$ and we convert the non-basis vector x_r as basis vector. Then divide the p^{th} row by $-a_{pj}$ to get,

   $$\frac{a_{p1}}{a_{pr}}x_1 + \frac{a_{p2}}{a_{pr}}x_2 + \dots + x_r + \dots + \frac{a_{pn}}{a_{pr}}x_n - \frac{1}{a_{pr}}s_p = \frac{b_p}{a_{pr}}.$$

Then relative cost of the variable x_j is given by, $c_j - c_r \frac{a_{pj}}{a_{pr}} \geq 0.$

$$\Rightarrow \quad \frac{c_j}{a_{pj}} \geq \frac{c_r}{a_{pr}} \quad \forall\, j = 1, 2, \ldots, n. \quad \Rightarrow \quad \frac{c_r}{-a_{pr}} \geq \frac{c_j}{-a_{pj}} \quad \forall\, j = 1, 2, \ldots, n.$$

$$\Rightarrow \quad \frac{c_r}{-a_{pr}} = \max_j \left\{ \frac{c_j}{-a_{pj}} \;;\; -a_{pj} < 0 \right\}.$$

4. Continue the process until we get a feasible solution.

Since the relative cost values are non-negative the solution becomes optimum when the values of the basis variables are non-negative ($\mathbf{V} \geq \mathbf{0}$). Then this feasible solution will be the optimum solution of the given problem (primal).

Example 9

Solve the lpp using dual simplex method,

Min $3x_1 + 5x_2 + 2x_3$

subject to $-x_1 + 2x_2 + 2x_3 \geq 3$,

$x_1 + 2x_2 + x_3 \geq 2$,

$-2x_1 - x_2 + 2x_3 \geq -4$,

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

1. **Find an initial basic infeasible solution**

Express the inequality constraints in the form of $\leq$ type inequality. Then we have,

Min $3x_1 + 5x_2 + 2x_3$

subject to $x_1 - 2x_2 - 2x_3 \leq -3$,

$-x_1 - 2x_2 - x_3 \leq -2$,

$2x_1 + x_2 - 2x_3 \leq 4$,

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

By introducing slack variables in the inequality constraints, we get

Min $3x_1 + 5x_2 + 2x_3 + 0s_1 + 0s_2 + 0s_3$

subject to $x_1 - 2x_2 - 2x_3 + s_1 = -3$,

$-x_1 - 2x_2 - x_3 + s_2 = -2$,

$2x_1 + x_2 - 2x_3 + s_3 = 4$,

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$.

Then, $s_1 = -3$, $s_2 = -2$, $s_3 = 4$ is the initial basic infeasible solution of the problem.

2. Simplex Iterations for optimum solution

C_B	X_B	V	3 **X1**	5 **X2**	2 **X3**	0 **S1**	0 **S2**	0 **S3**
0	S1	-3	1	-2	$\boxed{-2}$	1	0	0
0	S2	-2	-1	-2	-1	0	1	0
0	S3	4	2	1	-2	0	0	1
R.C			3	5	2	0	0	0
Max ratio $\frac{R.C}{a_{1j}}$; $a_{1j} < 0$			---	-2.5	-1	---	---	---

Here the relative cost values are non-negative so, we can proceed the iterations to find the optimum solution.

The most negative value in the basis is -3 hence remove s_1 from the basis. Then identify the maximum ratio $\frac{R.C}{a_{1j}}$; $a_{1j} < 0 = -1$. So, introduce x_3 to the basis. Then proceed to next simplex table.

C_B	X_B	V	3 **X1**	5 **X2**	2 **X3**	0 **S1**	0 **S2**	0 **S3**
2	X3	$3/2$	$-1/2$	1	1	$-1/2$	0	0
0	S2	$-1/2$	$-3/2$	-1	0	$\boxed{-1/2}$	1	0
0	S3	7	1	3	0	-1	0	1
R.C			4	3	0	1	0	0
Max ratio $\frac{R.C}{a_{2j}}$; $a_{2j} < 0$			$-8/3$	-3	---	-2	---	---
2	X3	2	1	2	1	0	-1	0
0	S1	1	3	2	0	1	-2	0
0	S3	8	4	5	0	0	-2	1
R.C			1	1	0	0	2	0

Since all entries in the value column (V) and relative cost (R.C) are non-negative, the optimum solution is obtained.

The optimum basis solution is, $x_3 = 2$, $s_1 = 1$, $s_3 = 8$ and hence the optimum value of the problem is 4 which occurs at $(x_1, x_2, x_3) = (0, 0, 2)$.