Integration

Integration are the reverse process of differentiation. If the differential coefficient of *f*(*x*) with respect *x* is $\varphi(x)$ then integral of $\varphi(x)$ with respect to *x* is *f*(*x*). That is if $\frac{df(x)}{dx} = \varphi(x)$ then $\int \varphi(x)dx = f(x)$. So integral is sometimes describe as anti-derivative.

Example: Let $f(x) = x^3$. Then $\frac{df(x)}{dx} = \frac{dx^3}{dx} = 3x^2$ then $\int 3x^2 dx = x^3$

The process of finding integral is called Integration.

Constant of Integration

Let $f(x) = x^3 + 10$. Then $\frac{df(x)}{dx} = \frac{d(x^3+10)}{dx} = 3x^2$ then $\int 3x^2 dx = 3x^3$ not equal to $x^3 + 10$. So in order to compensate this we add a constant "C" on the RHS of the integral. In general, we have $\int \varphi(x)dx = f(x) + C$. Here *C* is an arbitrary constant known as the constant of Integration.

Tables of basic Integrals

1. $\int x^n \, dx = \frac{1}{n+1} x^{n+1}, n \neq 1$
2. $\int \frac{1}{x} dx = \ln|x|$
3. $\int e^x \, dx = e^x$
4. $\int a^x \, dx = \frac{1}{\ln a} a^x$
5. $\int \ln x dx = x \ln x - x$
6. $\int \sqrt{x} \, dx = \frac{2}{3} x^{3/2}$
7. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$
8. $\int \frac{1}{ax \pm b} dx = \frac{1}{a} \ln(ax \pm b)$

Ruel's of Integration

Theorem 1: $\int kf(x)dx = k \int f(x) \, dx$ where *k* is a constant

Theorem 2: $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

The following table gives a comparison of Basic differentiation and integration rules.

Basic Integration Rules

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\, dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\, dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\, dx = k\int f(x)\, dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ Power Rule

Method of Integration.

1) Decomposition method

This method is based on the above two theorems.

Example 1: Integrate $4x^3 + 9x^2 + 6x + 4$

$$\int 4x^3 + 9x^2 + 6x + 4\, dx = 4\int x^3 dx + 9\int x^2 + 6\int x dx + 4\int dx$$

$$= 4\frac{x^{3+1}}{3+1} + 9\frac{x^{2+1}}{2+1} + 6\frac{x^{1+1}}{1+1} + 4x + C$$

$$= 4\frac{x^4}{4} + 9\frac{x^3}{3} + 6\frac{x^2}{2} + 4x + C$$

$$= x^4 + 3x^3 + 3x^2 + 4x + C$$

Example 2: Integrate $\frac{4x^4-3x^3+6x^2+4}{x}$

$$\int \frac{4x^4 - 3x^3 + 96 + 4}{x} dx = \int 4x^3 - 3x^2 + 6x + \frac{4}{x}\, dx$$

$$= 4\int x^3 dx - 3\int x^2 + 6\int x dx + 4\int \frac{1}{x} dx$$

$$= x^4 - x^3 + 3x^2 + 4\ln|x| + C.$$

2) Substitution method

For integration of a function of a function, take the intermediate function as u. Convert the given integrand into a function of u. Also express dx in terms of du. Then integrate. Now the answer is obtained in terms of u. Change u to x finally.

Example 3: Integrate $(3x+1)^2$

Let $u = 3x + 1$. Then $x = \frac{u-1}{3}$ and $\frac{du}{dx} = 3$. So $du = 3dx$ or $dx = \frac{1}{3}du$

$$\int (3x+1)^2 dx = \int u^2 \frac{1}{3} du = \frac{1}{3}\frac{u^3}{3} + C = \frac{1}{9}(3x+1)^3 + C$$

Example 4: Integrate $\sqrt{ax+b}$

Let $u = ax + b$. Then $\frac{du}{dx} = a$. or $dx = \frac{1}{a}du$. So

$\int \sqrt{ax+b}\, dx = \int \sqrt{u} \frac{1}{a} du = \frac{1}{a}\frac{2}{3}u^{3/2} + C$=$\frac{2}{3a}(ax+b)^{3/2}$+C.

Example 5: Integrate $\frac{1}{(4+3x)^5}$

Let $u = 4 + 3x$. Then $\frac{du}{dx} = 3$. or $dx = \frac{1}{3}du$. So

$$\int \frac{1}{(4+3x)^5} dx = \int \frac{1}{3u^5} du = \frac{1}{3}\int u^{-5} du = \frac{1}{3}\frac{u^{-5+1}}{(-5+1)} = \frac{-1}{12u^4}$$

$$= \frac{-1}{12(4+3x)^4} + C$$

Example 5: Integrate $\frac{4x+5}{2x^2+5x+2}$

Let $u = 2x^2 + 5x + 2$. Then $\frac{du}{dx} = 4x + 5$. or $du = (4x+5)dx$. So

$$\int \frac{4x+5}{2x^2+5x+2} dx = \int \frac{1}{u} du = \ln u + c = \ln(2x^2+5x+2) + c$$

Example 6: Integrate $\frac{\ln x}{x}$

Let $u = \ln x$ 2. Then $\frac{du}{dx} = \frac{1}{x}$. or $du = \frac{1}{x}dx$. So

$\int \frac{\ln x}{x} dx = \int u du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$

3) Integration by parts method

If *u* and *v* are two functions of *x* then the integral of the product *u* and *v* is given by

$$\int uvdx = u\int vdx - \int \left[\frac{du}{dx}\left(\int vdx\right)\right]dx$$

Example 7: Integrate xe^x

$\int xe^x dx = x\int e^x dx - \int \left[\frac{dx}{dx}(\int e^x dx)\right]dx$

$= xe^x - \int e^x dx = xe^x - e^x + C = (x-1)e^x + C$

Example 8: Integrate $\ln x$

$$\int \ln x\, dx = \int \ln x\,(1)dx = \ln x\int dx - \int \left[\frac{d\ln x}{dx}(\int dx)\right]dx$$

$$= (\ln x)x - \int \frac{1}{x}xdx = x\ln x - x + C$$

4) Integration using partial fraction method

Here we write $\frac{px+q}{(x-a)(x-b)}$ in the form $\frac{A}{(x-a)} + \frac{B}{(x-b)}$ which is called resolving into partial fraction.

Example 9: Integrate $\frac{x}{(x-1)(2x+1)}$

Writing $\frac{x}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$ (1)

Now multiplying both side with

$(x-1)(2x+1)$ we get

$$A(2x+1) + B(x-1) = x$$

Putting x=1 we get 3A=1 or $A = \frac{1}{3}$

Putting $x = \frac{-1}{2}$ we get $B = \frac{1}{3}$

So (1) can be written as $\frac{x}{(x-1)(2x+1)} = \frac{1/3}{x-1} + \frac{1/3}{2x+1}$

and $\int \frac{x}{(x-1)(2x+1)}dx = \int \frac{1/3}{x-1}dx + \int \frac{1/3}{2x+1}dx$

Let $u = x - 1$ then $dx = du$ and

$$\int \frac{1/3}{x-1} dx = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln u = \frac{1}{3} \ln(x-1)$$

Similarly, by taking $v = 2x + 1$ we get $dx = \frac{1}{2} du$

$$\int \frac{1/3}{2x+1} dx == \frac{1}{3} \int \frac{1}{2v} dv = \frac{1}{6} \ln v = \frac{1}{6} \ln(2x+1)$$

Hence

$$\int \frac{x}{(x-1)(2x+1)} dx = \frac{1}{3} \ln(x-1) + \frac{1}{6} \ln(2x+1)$$

Example 10: Integrate $\frac{x+12}{x^2-13x+42}$

$$x^2 - 13x + 42 = (x-6)(x-7)$$

$$\frac{x+12}{x^2-13x+42} = \frac{x+12}{(x-6)(x-7)} = \frac{A}{x-6} + \frac{B}{x-7} \qquad (1)$$

Multiplying through out with $= (x-6)(x-7)$ we get

$$x + 12 = A(x-7) + B(x-6)$$

Putin x=7 we get B =19

Putting x=6 we get A = -18 So

$$\int \frac{x+12}{x^2 - 13x + 42} dx = \int \frac{-18}{x-6} dx + \int \frac{19}{x-7} dx$$

= -18 log (x-6)+19 log(x-7)+C

Definite integral

Definite integral of $\varphi(x)$ over the limit (a,b) is the difference between the values of the integrand obtained when a and b are substituted in the integral of $\varphi(x)$ and is denoted by $\int_a^b \varphi(x)dx$. Ie if $\int \varphi(x)dx = f(x)$, then $\int_a^b \varphi(x)dx = f(b) - f(a)$. Actually the $\int_a^b \varphi(x)dx$ gives the are under the curve between a and b. Since both $f(b)$ and $f(a)$ contain C the difference does not contain C as it vanishes.

Example 11: Evaluate $\int_{-1}^{1}(3x^2-4x^3)dx$ (here we use substitution method)

$\int_{-1}^{1}(3x^2-4x^3)dx = 3\int_{-1}^{1}x^2dx - 4\int_{-1}^{1}x^3dx$ = $3\left[\frac{x^3}{3}\right]_{-1}^{1} - 4\left[\frac{x^4}{4}\right]_{-1}^{1}$

$= [1^3-(-1)^3][1^4-(-1)^4]$ = (1+1)- (1-1)=2.

Example 12: Evaluate $\int_1^2 x\sqrt{1+x^2}dx$

Let $u = 1+x^2$, then $du = 2xdx$ or $xdx = \frac{1}{2}du$. No when x=1, u= 2 and when x=2, u= 5

$\int_1^2 x\sqrt{1+x^2}dx = \int_2^5 \sqrt{u}\frac{1}{2}du$ =$\frac{1}{2}\times\frac{2}{3}\left[u^{3/2}\right]_2^5$= $\frac{1}{3}\left[5^{3/2}-2^{3/2}\right]$= 2.784

Example 13: Evaluate $\int_0^1 xe^x dx$ (here we use integration by part method)

$$\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = [xe^x]_0^1 - [e^x]_0^1 = (e-0)-(e-1) = 1$$

Example 14: Evaluate $\int_2^4 \frac{1}{x(x-1)}dx$ (here we use integration by partial fraction method)

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

So $A(x-1)+Bx = 1$ which gives A=-1 and B=1.

$$\int_2^4 \frac{1}{x(x-1)}dx = \int_2^4 \frac{-1}{x}dx + \int_2^4 \frac{1}{x-1}dx = [-\ln x]_2^4 - [\ln(x-1)]_2^4$$

$$= -\ln 4 + \ln 2 - \ln 3 + \ln 1.$$

Properties of Definite Integral

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$
3. $\int_a^b f(x)dx = \int_a^b f(y)dy$
4. If f(x) is an even function then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$
5. If f(x) is an odd function then $\int_{-a}^{a} f(x)dx = 0$.

Application of Integration in Economic

1. Marginal Revenue is the differential coefficient of Total Revenue. So the Total Revenue is the integral of the Marginal Revenue. That is
   Total Revenue (TR) = $\int MR\ dx$
2. Marginal cost is the differential coefficient of Total cost. So the Total cost is the integral of the Marginal cost. That is
   Total Cost (TC) = $\int MC\ dx$
3. Consumer's Surplus = $\int_0^Q f(q)dq - PQ$ where $f(q)$ is the demand function expressed in terms of q. Q stands for equilibrium quantity of demand and P stands for equilibrium price.
4. Producer's Surplus = $PQ - \int_0^Q g(q)dq$ where $g(q)$ is the supply function expressed in terms of q. Q stands for equilibrium quantity of supply and P stands for equilibrium price.

Example 15: Marginal revenue function for some product is 100-8q where q is the quantity sold. Calculate the total revenue when the demand for the product, q=10.

Given MR= 100-8q So

Total Revenue (TR) = $\int MR\ dq = \int (100 - 8q)dq = 100q - 4q^2 + C$

When q=0 we have TR = 0

i.e $0 = 100 \times 0 - 4 \times 0 + C \to C = 0$

So Total Revenue = $100q - 4q^2$

When q=10, TR =$100 \times 10 - 4 \times 100 = 1000 - 400 = 600.$

Example 16: Marginal cost function for some product is $1 + x + 6x^2$, where x is the output. Find the total cost function if the fixed cost is Rs.100 when the output is 0.

Given MC=$1 + x + 6x^2$

Total Cost (TC) = $\int MC\ dx$= $\int (1 + x + 6x^2)dx = x + \frac{x^2}{2} + 2x^3 + C$

Given TC =100 when x = 0. Therefore 0+0+2x0+C=100 → $C = 100$

Hence Total Cost (TC) =$x + \frac{x^2}{2} + 2x^3 + 100.$

Example 17: if the Marginal revenue function for out put q is given by $MR = \frac{6}{(q+2)^2} - 5.$ Find the demand function

$$MR = \frac{6}{(q+2)^2} - 5$$

Total Revenue (TR) = $\int MR\, dq = \int \left[\frac{6}{(q+2)^2} - 5\right] dq = 6\frac{(q+2)^{-2+1}}{-2+1} - 5q + C$

$$= \frac{-6}{(q+2)} - 5q + C$$

We know that when q=0 Tris also zero this gives

$\frac{-6}{(0+2)} - 5 \times 0 + C = 0 \rightarrow -3 + C = 0\ or$ C=3.

Total Revenue (TR) == $\frac{-6}{(q+2)} - 5q + 3 = \frac{-6+3(q+2)}{(q+2)} - 5q = \frac{3q}{(q+2)} - 5q$

Demand function p = $\frac{\text{Total Revenue}}{\text{Quantity Sold}} = \frac{TR}{q} = \frac{\frac{3q}{(q+2)} - 5q}{q} = \frac{3}{q+2} - 5.$

Example 18: if the demand law of a certain product is $p = 25 - 2q$. Calculate the consumer’s surplus when the equilibrium price for the product is Rs.5.

Given the demand function $p = 25 - 2q$ or $f(q) = 25 - 2q$

When p = 5, we have $5 = 25 - 2q \rightarrow q = 10.$ So Equilibrium quantity is Q =10 and $PQ = 5 \times 10 = 50$

Consumer’s Surplus = $\int_0^Q f(q)dq - PQ = \int_0^{10} (25 - q)dq - 50$

$$= 25[q]_0^{10} - [q^2]_0^{10} - 50 \text{ =250-100-50 =100.}$$

Example 19: Find the producer’s surplus when the supply function is given by

$p = 10 + 2q$ and the equilibrium price for the product is Rs.20.

Supply function is $p = 10 + 2q$ so $g(q) = 10 + 2q$

When p = 20, we have $20 = 10 + 2q \rightarrow q = 5$. So Equilibrium quantity is Q =5 and $PQ = 5 \times 20 = 100$

Producer's Surplus = $PQ - \int_0^Q g(q)dq = 100 - \int_0^5 [10 + 2q]dq$

$$= 100 - 10[q]_0^5 - [q^2]_0^5 \text{ =100-50-25 =25.}$$

Example 20: If $D = 250 - 50p$ and $S = 25p + 25$ are demand and supply functions, calculate the equilibrium price. Find consumers and producers surplus

For equilibrium D=S i.e in equilibrium we have $250 - 50p = 25p + 25$

Which gives $25p + 50p = 250 - 25 \rightarrow 75p = 225\ or\ p = 3$

∴ Equilibrium price, P=3

Consumer's Surplus

Given the demand function $q = 250 - 50p \rightarrow p = 5 - \frac{q}{50} = 5 - 0.02q$

So $f(q) = 5 - 0.02q$

When p = 3, we have $3 = 5 - 0.02 \times 3 \rightarrow q = 100$.

So Equilibrium quantity is Q =100 and $PQ = 3 \times 10 = 30$

Consumer's Surplus = $\int_0^Q f(q)dq - PQ = \int_0^{100} (5 - 0.02q)dq - 300$

$$= 5[q]_0^{100} - 0.01[q^2]_0^{100} - 300$$

Producer's surplus

the supply function is given by $S = 25p + 25$. That is the quantity of supply is $q = 25p + 25$ and the equilibrium price for the product is Rs.3.

So we have $25p = q - 25 \rightarrow p = \frac{q}{25} - 1 = 0.04q - 1$, $g(q) = 0.04q - 1$

When p=3, $3 = 0.04q - 1 \rightarrow q = 100$

So Equilibrium quantity is Q =100 and $PQ = 3 \times 100 = 300$

Producer's Surplus = $PQ - \int_0^Q g(q)dq = 300 - \int_0^{100} [0.04q - 1]dq$

$$= 300 - 0.04[q^2]_0^{100} + [q]_0^{100} \text{ =300-200+100 =200.}$$

Beta and Gamma function

Beta and gamma are the two most popular functions in mathematics. Gamma is a single variable function, whereas Beta is a two-variable function. The relation between beta and gamma function will help to solve many problems in physics and mathematics.

Gamma function

The Gamma function is important as it is an extension or generalization to the factorial function f (n) = n! where n is any positive integer to x!, where x is any real number.
The Gamma function is defined as the single variable function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Properties

1. The gamma function is convergent for x>0
2. $\Gamma(x+1) = x\Gamma(x)$
3. If x = n an integer then $\Gamma(n+1) = n!$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
5. $\Gamma(1) = 1$

Beta function

The Beta Function is defined as the two variable function as

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt$$

where $x \geq 1, y \geq 1$

Relation between beta and gamma functions

The connection between the beta function and the gamma function is given by

$$\boldsymbol{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}}$$

Improper integrals- beta and gamma

Improper integrals are definite integrals that cover an unbounded area. For example, $\int_1^\infty \frac{1}{x^2} dx$

An improper integral is a definite integral that has either one or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration. Hence improper integrals are of two type.

Type-1: (Infinite limits of integration): In this kind of integrals one or both limits of integration are infinity. For example, $\int_1^\infty \frac{1}{x^2} dx$, is an improper integral.

It can be viewed as the limit $\lim_{b\to\infty} \int_1^b \frac{1}{x^2} dx$.

Type-2: (Discontinuous integrand or integrands with vertical asymptotes): In this type of improper integrals the endpoints are finite, but the integrand function is unbounded at one (or two) of the endpoints. For example, $\int_0^1 \frac{1}{\sqrt{x}} dx$

It can be viewed as the limit $\lim_{a\to 0+} \int_a^1 \frac{1}{\sqrt{x}} dx$

Remark

Not all improper integrals have a finite value, but some of them definitely do. When the limit exists we say the integral is convergent, and when it doesn't we say it's divergent.

Gamma Integral.

An Integral of the form $\int_0^\infty t^{p-1} e^{-at} dt$ is called the gamma integral and its value is given by $\int_0^\infty t^{p-1} e^{-at} dt = \frac{\Gamma(p)}{a^p}$

Beta Integral.

An Integral of the form $\int_0^1 t^{p-1} (1-t)^{q-1} dt$ is called the beta integral and its value is given by $\int_0^1 t^{p-1} (1-t)^{q-1} dt = B(p,q)$

Difference and Differential equation

Difference equation, mathematical equality involving the differences between successive values of a function of a discrete variable. A discrete variable is one that is defined or of interest only for values that differ by some finite amount, usually a constant and often 1; for example, the discrete variable x may have the values $x_0 = a$, $x_1 = a + 1$, $x_2 = a + 2$, . . ., $x_n = a + n$. The function y has the corresponding values y_0, y_1, y_2, . . ., y_n, from which the differences can be found:

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\dots.$$

$$\Delta y_n = y_{n+1} - y_n$$

Any equation that relates the values of Δy_i to each other or to x_i is a difference equation. In general, such an equation takes the form $y_i - a_i y_{i-1} = b_i$

Systematic methods have been developed for the solution of these equations and for those in which, for example, second-order differences are involved. A second-order difference is defined as

$$\begin{aligned} \Delta^2 y_i &= \Delta(\Delta y_i) = \Delta y_{i+1} - \Delta y_i \\ &= (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) \\ &= y_{i+2} - 2y_{i+1} + y_i \end{aligned}$$

Like this we can derive any order diffrence equation.

Differential equation

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y=f(x)$ and its derivative, known as a differential equation.

A *differential equation* is an equation involving an unknown function *y*=*f*(*x*) and one or more of its derivatives. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

Example

1. $\frac{dy}{dx} = 3x^2 + 2x + 4$
2. $x\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2e^{-5y}$
3. $ay" + by' + cy = g(x)$
4. $y^{(4)} + 10y^{(3)} - 4y^{(2)} + 2y = \ln x$
5. $\alpha^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x}$
6. $\frac{\partial^2 u}{\partial x \partial y} = 1 + \frac{\partial y}{\partial x}$

Order of the differential equation.

The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above (1) is a first order differential equation, (2) (3) (5) and (6) are second order differential equations, (4) is a is a fourth order differential equation.

Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

Ordinary and Partial Differential Equations

A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has partial derivatives in it. In the differential equations above (1), (2), (3) and (4) are ode's and (5) and (6) are pde's.

Linear differential equation

A **linear differential equation** is any differential equation that can be written in the following form.

The important thing to note about linear differential equations is that there are no products of the function, y(t), and its derivatives and neither the function or its derivatives occur to any power other than the first power. Also note that neither the function or its derivatives are “inside” another function, for example, $\sqrt{y'}$, e^y

$$a_n(t)\,y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t)$$

.

The coefficients a0(t),…,an(t) and g(t) can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, y(t), and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the above form, then it is called a **non-linear** differential equation. In the above (1), (3) and (4) are linear equations, (2) is a non linear equation.

Solution

A **solution** to a differential equation on an interval $\alpha<t<\beta$

is any function y(t) which satisfies the differential equation in question on the interval $\alpha<t<\beta$. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution.