Operations Research

During World War II, the military management in England called on a team of scientist to study the strategic and technical problem of air and land defence. This team of scientists included physicists, mathematicians, statisticians, engineers, psychologists and many others. The objective was to determine the most effective utilization of limited military resources. This group of scientists forms the first OR team. The name operations research (or operational research) was apparently coined because the team was carrying out research on military operations. Immediately after the war, the success of military team attracted the attention of industrial managers who were seeking solution to their problem. As a result OR soon spread from military to Government, industrial, social and economic planning.

Definition of Operations Research

Operations research is a scientific methods, techniques and tools of providing executive department with a quantitative basis for decision regarding the operations under their control.

Characteristic of Operations Research

The essential characteristic of operations research are

1. OR study the system as a whole. i.e. OR is system (or Executive) Orientation.
2. OR uses interdisciplinary teams
3. The application of Scientific Methods
4. OR cannot cover new problems arises in later stage

Phases of Operations Research

The different phases of the Operations Research approach are

1. Formulation the Problem
2. Construction of a model to represent the system under study
3. Deriving a solution from the model
4. Testing the model and the solution derived from it.
5. Establishing the control over solution.
6. Establishing the solution to work, i.e. implementation.

Models of Operations Research

The main models of Operations Research are

1. Mathematical Programming techniques
2. Inventory Models
3. Allocation Model
4. Transportation Problems
5. Sequencing models
6. Routing models
7. Competitive models
8. Queuing models
9. Dynamic Programming models
10. Simulation techniques
11. Decision theory
12. Replacement models
13. Heuristic models
14. Combined methods

Linear Programming Problem

The mathematical programming problems in general deals with determining optimal allocation of limited resources to meet given objectives. The resources may be materials, men, machines etc. The main branches of mathematical programming are Linear Programming (LPP), Non-Linear Programming (NLPP), Integer Programming (IPP) and Goal Programming. A Linear Programming Problem deals with the optimization (maximization or minimization) of a function of variables known as *Objective function*, subject to a set of linear equations and/or in equations known as restrictions or constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The restrictions may be imposed by different sources such as market demand, production processes and equipment, storage capacity, raw material availability etc. By linearity is meant a mathematical expression (equations) in which the variables do not have powers.

Thus Linear Programming may be defined as a method to obtain an optimum solution to a linear objective function subject to a set of linear constraints as defined above.

The requirement of a Linear Programming Problem

Linear programming problem can be used for optimization problems if the following conditions are satisfied

1. There must be a well defined objective function (in terms of profit, cost or quantity produced) which is to maximized or minimized and which can be expressed as a linear function of decision variables.
2. There must be restrictions on the amount or extent of attainment of the objective and these restrictions must be capable of being expressed as linear equalities or inequalities in terms of variables.
3. There must be alternative course of action. For example, given a product may be processed by two different machines and problem may be as to how much of the product to allocate to which machine.
4. Another necessary requirement is that the decision variables should be interrelated and non-negative. The non-negativity condition shows that linear programming deals with real life situations for which negative quantities are illogical.
5. Finley the resources must be in limited supply.

General form of a Linear Programming Problem

Optimize $Z = c_1x_1 + c_2x_2 + .. + c_nx_n$.

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = (\text{or} \geq \text{or} \leq) b_1.$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = (\text{or} \geq \text{or} \leq) b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = (\text{or} \geq \text{or} \leq) b_m.$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

Advantages of a Linear Programming Problem

1. It provides the insight and perspectives into problem environment related with a multi dimensional phenomenon. This generally results in clear picture of the true problem.
2. It makes a scientific and mathematical analysis of the problem situations. It also considers all possible aspects and remedies associates with the problem.
3. It gives an opportunity to the decision maker to formulate his strategies consistent with the constraints and the objectives.
4. It deals with changing situations. Once a plan is arrived through a Linear Programming, it can be revaluated for changing conditions.

Formulation of Linear Programming Problem.

The main steps in formulation of a Linear Programming Problem are

Steps

1. Study the situation and find the key-decision to be made. (i.e. to identify whether the problem is minimization or maximization)
2. Assume symbols for variable quantities or identify the decision variables noticed in step 1.
3. Express the feasible alternative mathematically in terms of variables. Feasible alternative are those which are physically, economically and financially possible.
4. Mention the objective quantity and expressed it as a linear function of the variables.
5. Express the influence factors or restriction (constrains) into mathematical linear functions.

Example –1

A firm produces three products. Three products are processed on three different machines. The time required to manufacture one unit of each of three products and the daily capacity of the tree machine are given in the table below

Machine	Time per unit (minutes)			Machine Capacity
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactures for each product. The profit per unit for product 1, 2 and 3 are Rs.4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the LPP

Step 1: The Key –decision to be made is to determine the number of each product 1, 2 and 3 to be produced so has to maximize profit.

Step 2: The decision variables are x_1, x_2 and x_3, which represent the number of units of product 1, 2 and 3 produced daily.

Step 3: The feasible alternatives are the set of variables x_1, x_2 and x_3, where $x_1, x_2, x_3, \geq 0$.

Step 4: The objective function is Maximize Z= $4x_1+3x_2+6x_3$

Step 5: The constrains are

$$2x_1+3x_2+2x_3 \leq 440$$
$$4x_1+0x_2+3x_3 \leq 470$$
$$2x_1+5x_2+0x_3 \leq 430.$$

Thus the LPP is

Maximize Z= $4x_1+3x_2+6x_3$

Subject to

$$2x_1+3x_2+2x_3 \leq 440$$
$$4x_1+3x_3 \leq 470$$
$$2x_1+5x_2 \leq 430.$$
$$x_1, x_2, x_3, \geq 0.$$

Example –2

An advertising company wishes to plan its advertising strategy in three different media-television, radio and magazines. The purpose of advertising is to reach as large a number of potential customers as possible. Following data has obtained from market survey

	Television	Radio	Magazine I	Magazine II
Cost of advertising units	Rs.30,000	Rs.20,000	Rs.15,000	Rs.10,000
No of potential customers reached per unit	2,00,000	6,00,000	1,50,000	1,00,000
No of female customers reached per unit	1,50,000	4,00,000	70,000	50,000

The company wants to spend mot more than Rs. 450,000 on advertising. Following are the further requirement that must be met:

1. at least 1 million exposures take place among female customers
2. advertising on magazine be limited to Rs.1,50,000
3. at least 3 advertising units be brought on magazine I and 1 units in magazine II and
4. the number of advertising units on television and radio should each be between 5 and 10.

Formulate the LPP

Step 1: The Key –decision to be made is to determine the number of advertising units to be brought in television, radio and magazines so has to reach as large a number of potential customers

Step 2: The decision variables are x_1, x_2, x_3 and x_4, which represent the number of advertising units in television, radio, magazine I and magazine II.

Step 3: The feasible alternatives are the set of variables x_1, x_2, x_3 and x_4, where $x_1, x_2, x_3, x_4 \geq 0$.

Step 4: The objective function is Maximize $Z= 10^5\ (2x_1+6x_2 +1.5\ x_3+1\ x_4)$

Step 5: The constrains are

$$30{,}000x_1+20{,}000x_2 +15{,}000x_3 +10{,}000x_4 \le 4{,}50{,}000$$
$$150{,}000x_1+400{,}000x_2 +70{,}000x_3 +50{,}000x_4 \ge 1{,}00{,}000$$
$$15{,}000x_3 +10{,}000x_4 \le 1{,}50{,}000$$
$$x_3 \ge 3$$
$$x_4 \ge 2$$
$$5 \le x_1 \le 10$$
$$5 \le x_2 \le 10$$

Thus the LPP is

$$\text{Maximize } Z= 10^5\ (2x_1+6x_2 +1.5\ x_3+1\ x_4)$$

Subject to

$$30x_1+20x_2 +15x_3 +10x_4 \le 450$$
$$15x_1+40x_2 +7x_3 +x_4 \ge 100$$
$$15x_3 +10x_4 \le 150$$
$$x_3 \ge 3$$
$$x_4 \ge 2$$
$$x_1 \ge 5$$
$$x_2 \ge 5$$
$$x_1 \le 10$$
$$x_2 \le 10$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Solution of Linear Programming Problem

In general there are two methods to solve the Linear Programming Problem. They are graphical methods if there are only two decision variables and the simplex method. Now we consider some definition related to this

Feasible Solution: A feasible solution to a Linear Programming Problem is the set of the values of the variables which satisfy all the.

Basic feasible solutions: A feasible solution is called a basic feasible solution if it has no more than m positive (non-zero) x_j. In other words, it is a basic solution which also satisfies the non-negativity condition.

Optimal Solution: A basic feasible solution to a Linear Programming Problem is said to be optimum solution if it optimizes the objective function of the problem.

Slack Variables: If the constrains has a $\le$ then in order to make it an equality we have to add some variable to the left hand side of the constraints. This variables are called slack variables.

Surplus Variables: If the constrains has a sign $\ge$ then in order to make it an equality we have to subtract some variable to the left hand side of the constraints. This variables are called surplus variables.

Graphical Method of Solution to Linear Programming Problem

So far we have learnt how to construct a mathematical model for a linear programming problem. If we can find the values of the decision variables x_1, x_2, x_3, ... x_n, which can optimize (maximize or minimize) the objective function Z, then we say that these values of x_i are the optimal solution of the Linear Program (LP).

The graphical method is applicable to solve the LPP involving two decision variables x_1, and x_2, we usually take these decision variables as x, y instead of x_1, x_2. To solve an LP, the graphical method includes two major steps.

Linear Programming Problem involving two variables can be solved by Graphical method. Since the solution has to satisfies the non negative conditions the value of the variable x_1 and x_2 can lie only in the first quadrant. The main steps for solving a LPP by graphic method are

1. Formulate the problem into a Linear Programming Problem
2. Each inequality in the constrains may be written as equality
3. Draw straight lines corresponding to the equations obtained in step 2. So there will be many straight line as the constraints are.
4. Identify the feasible region. Feasible region is the area which satisfies all the constrains simultaneously
5. The vertices of the feasible solutions are to be located and their co ordinates are to be measured.
6. Calculate the value of the objective function at each vertex.
7. The solution is the co-ordinate of the vertex, which optimizes the objective function, and the corresponding value of the objective function is the optimum value.

Example

Solve the following LPP

Maximize $Z = 3x_1 + 5x_2$

Subject to

$$2x_1 + 6x_2 \leq 36$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1, x_2, \geq 0.$$

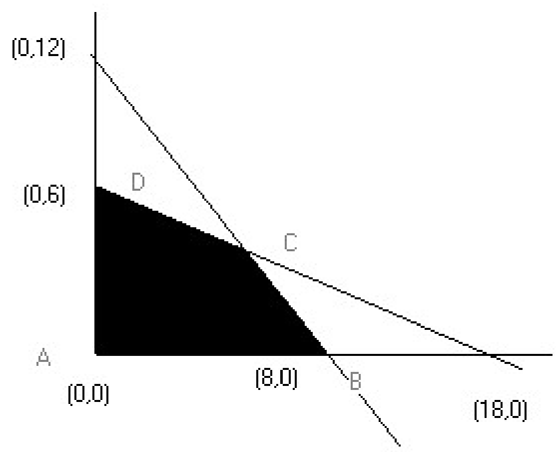
Change the constrains to equality we have

$$2x_1 + 6x_2 = 36 \qquad (1)$$
$$3x_1 + 2x_2 = 24 \qquad (2)$$

Now consider equation (1) ,when $x_1 = 0$, $x_2 = 6$ and when $x_2 = 0$, $x_1 = 18$, so the two point in line (1) are (0,6) and (18,0). Similarly for equation (2) we have two points (0,12) and (8,0). Plotting this we get the straight line graphs as

Vertex	Function value of Z
A=(0,0)	0
B=(8,0)	24
C=(36/7,60/7)	58.2857
D=(0,6)	30

So the optimum solution is (36/7,60/7) and the optimum value is 58.2857.



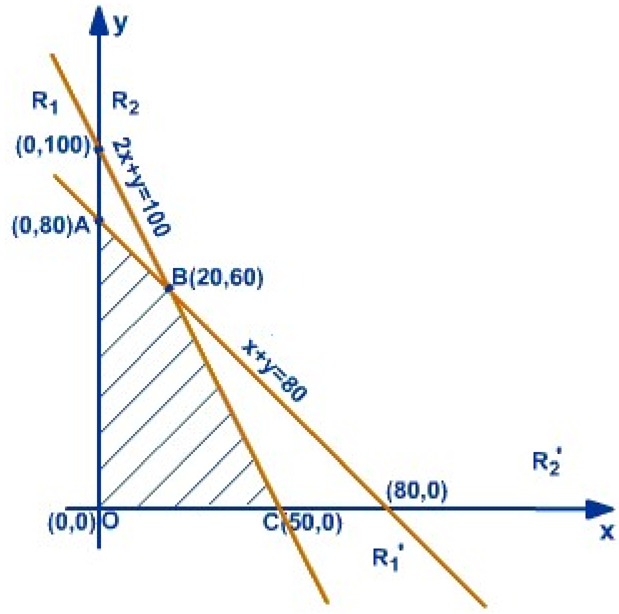
Example-2

Max Z = 50x + 18y

Subjected to

$2x+y\leq 100$
$x+y\leq 80$
$x, y\geq 0$

To determine two points on the straight line 2x + y = 100
Put y = 0, 2x = 100 $\Rightarrow$(50, 0) is a point on the line
put x = 0 in (2), y =100 $\Rightarrow$(0, 100) is a point on the line
To determine two points on the straight line x + y = 80
Put y = 0, $\Rightarrow$(80, 0) is a point on the line
put x = 0 $\Rightarrow$(0, 80) is a point on the line
Plotting these lines on the graph paper we have



The intersection of both the region R_1 and R_1' is the feasible solution of the LPP. Therefore every point in the shaded region OABC is a feasible solution of the LPP, since this point satisfies all the constraints including the non-negative constraints.

Vertex	Function value of Z
O(0,0)	0
A(0,80)	1440
B(20,60)	2080
C(50,0)	2500

So the optimum solution is (50,0) and the optimum value is 2500

Example-3

Minimize Z =100x + 100y.

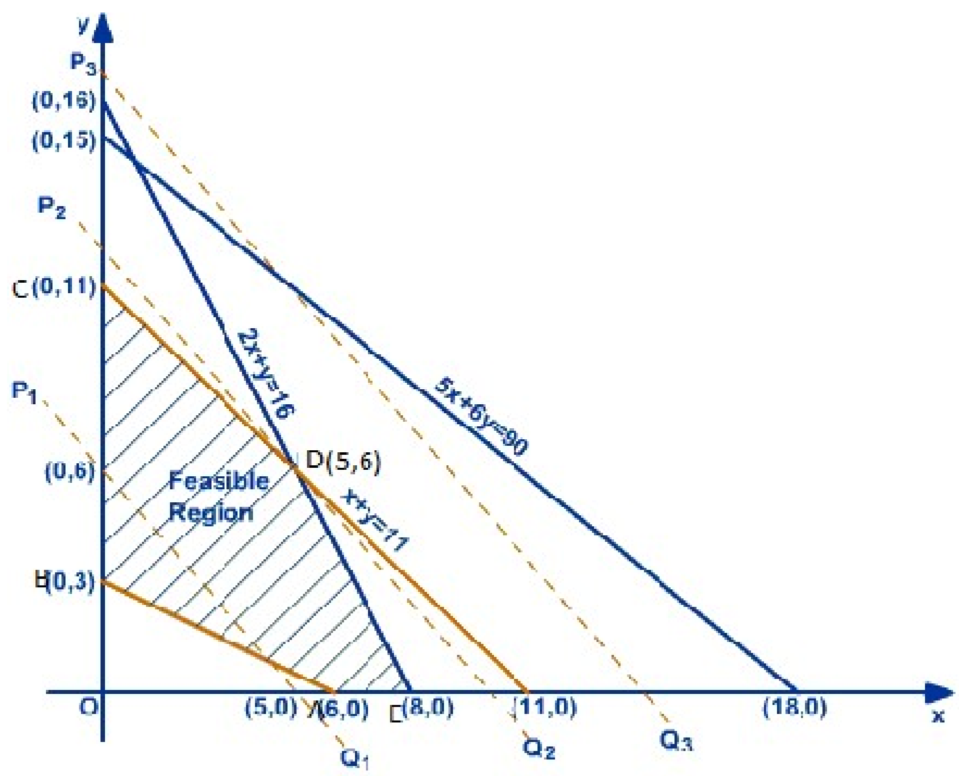
$10x+5y\leq 80$
$6x+6y\leq 66$
$4x+8y\geq 24$
$5x+6y\leq 90$
$x, y\geq 0$

since $x\geq 0$, $y\geq 0$, consider only the first quadrant of the plane graph the following straight lines on a graph paper

$10x + 5y = 80$ or $2x+y =16$
$6x + 6y = 66$ or $x +y =11$
$4x+ 8y = 24$ or $x+ 2y = 6$
$5x + 6y = 90$

Vertex	Function value of Z
A(0,6)	600
B(0,3)	300
C(0,11)	1100
D(5,6)	1100
E(8,0)	800

So the optimum solution is (0,300) and the optimum value is 300



SIMPLEX METHOD

Simplex method was originally developed by G.B.Dantzig, an American mathematician.

Simplex method is a Linear Programming technique in which we start with a certain solution which is feasible. We improve this solution in a number of consecutive stages until we arrive at an optimal solution.

For arriving at the solution of LPP by this method, the constraints and the objective function are presented in a table known as simplex table. Then following a set procedure and rules, the optimal solution is obtained making step by step improvement.

Thus Simplex method is an iterative (step by step) procedure in which we proceed in a systematic steps from an initial Basic Feasible Solution to another Basic Feasible Solution and finally, in a finite number of steps to an optimal basic feasible solution, in such a way that the value of the objective function at each step is better (or at least not worse) than that at the preceding steps. In other words the simplex algorithm consists of the following main steps

(1) Find a trial Basic Feasible Solution of the Linear Programming Problem.
(2) Test whether it is an optimal solution or not.
(3) If not optimal, improve the first trial Basic Feasible Solution by a set of rules. That is determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value
(4) Repeat the steps (2) and (3) till an optimal solution is obtained.

How to construct a simplex table?

Simplex table consists of rows and columns. If there are 'm' original variables and 'n' introduced variables, then there will be 3+m+n columns in the simplex table. (Introduced variables are slack, surplus or artificial variables).

First column (B) contains the basic variables. Second column (C) shows the coefficient of the basic variables in the objective function. Third column (x_B) gives the values of basic variables. Each of next 'm+n' columns contain coefficient of variables in the constraints, when they are converted into equations.

Basic (B)

The variables whose values are not restricted to zero in the current basic solution, are listed in one column of the simplex table known as Basis (B).

Basic variables

The variables which are listed in the basis are called basic variables and others are known as non-basic variables.

Vector

Any column or row of a simplex table is called a vector. So we have X_1 – vector, X_2 – vector etc.

In a simplex table, there is a vector associated with every variable. The vectors associated with the basic variables are unit vectors.

Unit vector

A vector with one element 1 and all other elements zero, is a unit vector.

Eg: $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix}$ are unit vectors.

Net Evaluation (Δ_j)

Δ_j is the net profit or loss if one unit of the variable in the respective column is introduced. That is, Δ_j shows what is the profit (or loss) if one unit of x_j is introduced. The row containing Δ_j values is called net evaluation row or index row.

$\Delta_j = C_j - Z_j$

where c_j is the coefficient of x_j variables in the objective function and z_j is the sum of the products of coefficients of basic variables in the objective function and the vector x_j.

Minimum Ratio

Minimum ratio is the lowest non negative ratio in the replacing ratio column.

The replacing ratio column (θ) contains values obtained by dividing each element in x_B column (the column showing the values of the basic variable) by the corresponding elements in the incoming vector.

Key Column (incoming vector)

The column which has highest negative Δ_j in a maximization problem or the highest positive Δ_j in a minimization problem, is called incoming vector.

Key row (outgoing vector)

The row which relates to the minimum ratio, is the outgoing vector.

Key element

Key element is that element of the simplex table which lies both in the key row and key column.

Iteration

Iteration means step by step process followed in simplex method to move from one basic feasible solution to another.

Computational procedure of simplex method (Simplex Algorithm)

Step1: Formulate the problem into a LPP

Step2: Convert the constraints into equations by introducing the non-negative slack variables or surplus variables wherever necessary.

Step3: Construct starting simplex table.

Step4: Conduct the test of optimality.

This is done by computing net evaluation $\Delta_j = C_j - Z_j$

The solution under test is not optimal if at least one Δ_j is positive for maximization case. If at least on Δ_j is negative, the solution is not optimal for minimization case. Otherwise the solution is optimal.

If solution under test is not optimal, we must proceed to the next step.

Step5: Find incoming and outgoing vectors.

The incoming vector corresponds to highest Δ_j for maximization cases and highest positive Δ_j for minimization cases. Outgoing vector corresponds to minimum ratio.

Step6: The element which is at the intersection of minimum ratio arrow (⟵) and incoming vector arrow (↑) is called the key element. We mark this element in

There should be 1 at the positon of key element. If it is not 1 then divide all the elements of the row, containing key element, by the key element. Then add appropriate multiples of the corresponding elements of this changed row to the elements of all other rows. Now obtain the next simplex table with the changes. Improved basic feasible solution can be readout from the simplex table. The solution is obtained by reading B – column and x_B column together.

Step7: Now test the above improved B. F. S. for optimality as in step 4.

If this solution is not optimal then repeat steps (5) and (6) until an optimal solution is finally obtained.

Artificial variable

Artificial variables are fictitious variables. They are incorporated only for computational purposes. They have no physical meaning. Artificial variables are introduced when the constraints are of the type ≥ or =.

Big – M Method

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent. When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

Big M method is a modified simplex method for solving a LPP when a high penalty cost (or profit) M has been assigned to the artificial variable in the objective function.

When artificial variables are introduced, we include these artificial variables in the basis (B) first. These artificial variables are driven out in the first iteration. For this purpose, we assign a very large M to teach artificial variables as coefficient in the objective function, The quantity 'M' is known as penalty. In maximization cases –M and in minimization cases +M are assigned to the artificial variables as their coefficients in objective functions.

Big M method can be applied to minimization problems as well as maximization problems.

Simplex Method with 'greater-than-equal-to' (ε) and equality (=) constraints

The LP problem, with 'greater-than-equal-to' (ε) and equality (=) constraints, is transformed to its standard form in the following way.

1. One 'artificial variable' is added to each of the 'greater-than-equal-to' (ε) and equality (=) constraints to ensure an initial basic feasible solution.
2. Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient *M* for maximization (minimization) problem.
3. Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.
4. If there are more than one artificial variable, step 3 is repeated for all the artificial variables one by one.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.
2. Convert each inequality constraint to standard form (If constraint *i* is a ≤ constraint, we add a slack variable *si*; and if constraint *i* is a ≥ constraint, we subtract an excess variable *ei*).
3. Add an artificial variable *ai* to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $ai \geq 0$.
4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) A*i* to the objective function. If the LP is a max problem, add (for each artificial variable) -M*ai* to the objective function.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem. If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Example-1

Max $Z = 7x_1 + 5x_2$

subject to

$x_1 + 2x_2 \le 6$

$4x_1 + 3x_2 \le 12$

and $x_1, x_2 \ge 0$;

Max $Z = 7x_1 + 5x_2 + 0S_1 + 0S_2$

subject to

$x_1 + 2x_2 + S_1 = 6$

$4x_1 + 3x_2 + S_2 = 12$

and $x_1, x_2, S_1, S_2 \ge 0$

Iteration-1		C_j	7	5	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio $\frac{X_B}{x_1}$
S_1	0	6	1	2	1	0	$\frac{6}{1} = 6$
S_2	0	12	(4)	3	0	1	$\frac{12}{4} = 3 \rightarrow$
$Z = 0$		Z_j	**0**	**0**	**0**	**0**	
		$C_j - Z_j$	7↑	5	0	0	

Positive maximum $C_j - Z_j$ is 7 and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 3 and its row index is 2. So, the leaving basis variable is S_2.

∴ The pivot element is 4.

Entering $= x_1$, Departing $= S_2$, Key Element $= 4$

Iteration-2		C_j	7	5	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	MinRatio
S_1	0	3	0	$\frac{5}{4}$	1	$-\frac{1}{4}$	
x_1	7	3	1	$\frac{3}{4}$	0	$\frac{1}{4}$	
$Z = 21$		Z_j	**7**	$\mathbf{\frac{21}{4}}$	**0**	$\mathbf{\frac{7}{4}}$	
		$C_j - Z_j$	0	$-\frac{1}{4}$	0	$-\frac{7}{4}$	

Since all $C_j - Z_j \le 0$

Hence, optimal solution is arrived with value of variables as :
$x_1 = 3, x_2 = 0$

Max $Z = 21$

Example-2

Max $Z = 5x_1 + 3x_2$

subject to

$x_1 + x_2 \le 2$

$5x_1 + 2x_2 \le 10$

$3x_1 + 8x_2 \le 12$

and $x_1, x_2 \ge 0$;

Max $Z = 5x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3$

subject to

$x_1 + x_2 + S_1 = 2$

$5x_1 + 2x_2 + S_2 = 10$

$3x_1 + 8x_2 + S_3 = 12$

and $x_1, x_2, S_1, S_2, S_3 \ge 0$

Iteration-1		C_j	5	3	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	**MinRatio** $\frac{X_B}{x_1}$
S_1	0	2	(1)	1	1	0	0	$\frac{2}{1} = 2 \rightarrow$
S_2	0	10	5	2	0	1	0	$\frac{10}{5} = 2$
S_3	0	12	3	8	0	0	1	$\frac{12}{3} = 4$
$Z = 0$		Z_j	**0**	**0**	**0**	**0**	**0**	
		$C_j - Z_j$	$5 \uparrow$	3	0	0	0	

Positive maximum $C_j - Z_j$ is 5 and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 2 and its row index is 1. So, the leaving basis variable is S_1.

$\therefore$ The pivot element is 1.

Entering $= x_1$, Departing $= S_1$, Key Element $= 1$

Iteration-2		C_j	5	3	0	0	0	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	**MinRatio**
x_1	5	2	1	1	1	0	0	
S_2	0	0	0	-3	-5	1	0	
S_3	0	6	0	5	-3	0	1	
$Z = 10$		Z_j	**5**	**5**	**5**	**0**	**0**	
		$C_j - Z_j$	0	-2	-5	0	0	

Since all $C_j - Z_j \le 0$

Hence, optimal solution is arrived with value of variables as :
$x_1 = 2, x_2 = 0$

Max $Z = 10$

Example-3

Max $Z = 3x_1 + 5x_2 + 4x_3$

subject to

$2x_1 + 3x_2 \leq 8$

$2x_2 + 5x_3 \leq 10$

$3x_1 + 2x_2 + 4x_3 \leq 15$

and $x_1, x_2, x_3 \geq 0$;

After introducing slack variables

Max $Z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$

subject to

$2x_1 + 3x_2 + S_1 = 8$

$2x_2 + 5x_3 + S_2 = 10$

$3x_1 + 2x_2 + 4x_3 + S_3 = 15$

and $x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Iteration-1		C_j	3	5	4	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	**MinRatio** $\frac{X_B}{x_2}$
S_1	0	8	2	(3)	0	1	0	0	$\frac{8}{3} = 2.6667 \rightarrow$
S_2	0	10	0	2	5	0	1	0	$\frac{10}{2} = 5$
S_3	0	15	3	2	4	0	0	1	$\frac{15}{2} = 7.5$
$Z = 0$		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	3	$5\uparrow$	4	0	0	0	

Positive maximum $C_j - Z_j$ is 5 and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 2.6667 and its row index is 1. So, the leaving basis variable is S_1.

$\therefore$ The pivot element is 3.

Iteration-2		C_j	3	5	4	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	**MinRatio** $\frac{X_B}{x_3}$
x_2	5	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	---
S_2	0	$\frac{14}{3}$	$-\frac{4}{3}$	0	(5)	$-\frac{2}{3}$	1	0	$\frac{\frac{14}{3}}{5} = \frac{14}{15} = 0.9333 \rightarrow$
S_3	0	$\frac{29}{3}$	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	$\frac{\frac{29}{3}}{4} = \frac{29}{12} = 2.4167$
$Z = \frac{40}{3}$		Z_j	$\frac{10}{3}$	5	0	$\frac{5}{3}$	0	0	
		$C_j - Z_j$	$-\frac{1}{3}$	0	$4\uparrow$	$-\frac{5}{3}$	0	0	

Positive maximum $C_j - Z_j$ is 4 and its column index is 3. So, the entering variable is x_3.

Minimum ratio is 0.9333 and its row index is 2. So, the leaving basis variable is S_2.

$\therefore$ The pivot element is 5.

Iteration-3		C_j	3	5	4	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	**MinRatio** $\frac{X_B}{x_1}$
x_2	5	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{\frac{8}{3}}{\frac{2}{3}} = 4$
x_3	4	$\frac{14}{15}$	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	---
S_3	0	$\frac{89}{15}$	$\left(\frac{41}{15}\right)$	0	0	$-\frac{2}{15}$	$-\frac{4}{5}$	1	$\frac{\frac{89}{15}}{\frac{41}{15}} = \frac{89}{41} = 2.1707 \rightarrow$
$Z = \frac{256}{15}$		Z_j	$\frac{34}{15}$	5	4	$\frac{17}{15}$	$\frac{4}{5}$	0	
		$C_j - Z_j$	$\frac{11}{15} \uparrow$	0	0	$-\frac{17}{15}$	$-\frac{4}{5}$	0	

Positive maximum $C_j - Z_j$ is $\frac{11}{15}$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 2.1707 and its row index is 3. So, the leaving basis variable is S_3.

$\therefore$ The pivot element is $\frac{41}{15}$.

Iteration-4		C_j	3	5	4	0	0	0
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_2	5	$\frac{50}{41}$	0	1	0	$\frac{15}{41}$	$\frac{8}{41}$	$-\frac{10}{41}$
x_3	4	$\frac{62}{41}$	0	0	1	$-\frac{6}{41}$	$\frac{5}{41}$	$\frac{4}{41}$
x_1	3	$\frac{89}{41}$	1	0	0	$-\frac{2}{41}$	$-\frac{12}{41}$	$\frac{15}{41}$
$Z = \frac{765}{41}$		Z_j	3	5	4	$\frac{45}{41}$	$\frac{24}{41}$	$\frac{11}{41}$
		$C_j - Z_j$	0	0	0	$-\frac{45}{41}$	$-\frac{24}{41}$	$-\frac{11}{41}$

Since all $C_j - Z_j \leq 0$

Hence, optimal solution is arrived with value of variables as :
$x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}$

Max $Z = \frac{765}{41}$

Example-4 (BIG M method)

Min $Z = 5x_1 + 6x_2$

subject to

$2x_1 + 5x_2 \geq 1500$

$3x_1 + x_2 \geq 1200$

and $x_1, x_2 \geq 0$;

After introducing surplus,artificial variables

Min $Z = 5x_1 + 6x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

subject to

$2x_1 + 5x_2 - S_1 + A_1 = 1500$

$3x_1 + x_2 - S_2 + A_2 = 1200$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Iteration-1		C_j	5	6	0	0	M	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	MinRatio $\frac{X_B}{x_2}$
A_1	M	1500	2	(5)	-1	0	1	0	$\frac{1500}{5} = 300 \rightarrow$
A_2	M	1200	3	1	0	-1	0	1	$\frac{1200}{1} = 1200$
$Z = 2700M$		Z_j	$5M$	$6M$	$-M$	$-M$	M	M	
		$C_j - Z_j$	$-5M + 5$	$-6M + 6 \uparrow$	M	M	0	0	

Negative minimum $C_j - Z_j$ is $-6M + 6$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 300 and its row index is 1. So, the leaving basis variable is A_1.

$\therefore$ The pivot element is 5.

Iteration-2		C_j	5	6	0	0	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	A_2	MinRatio $\frac{X_B}{x_1}$
x_2	6	300	$\frac{2}{5}$	1	$-\frac{1}{5}$	0	0	$\frac{300}{\frac{2}{5}} = 750$
A_2	M	900	$\left(\frac{13}{5}\right)$	0	$\frac{1}{5}$	-1	1	$\frac{900}{\frac{13}{5}} = \frac{4500}{13} = 346.1538 \rightarrow$
$Z = 900M + 1800$		Z_j	$\frac{13M}{5} + \frac{12}{5}$	6	$\frac{M}{5} - \frac{6}{5}$	$-M$	M	
		$C_j - Z_j$	$-\frac{13M}{5} + \frac{13}{5} \uparrow$	0	$-\frac{M}{5} + \frac{6}{5}$	M	0	

Negative minimum $C_j - Z_j$ is $-\frac{13M}{5} + \frac{13}{5}$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 346.1538 and its row index is 2. So, the leaving basis variable is A_2.

$\therefore$ The pivot element is $\frac{13}{5}$.

Iteration-3		C_j	5	6	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**
x_2	6	$\frac{2100}{13}$	0	1	$-\frac{3}{13}$	$\frac{2}{13}$
x_1	5	$\frac{4500}{13}$	1	0	$\frac{1}{13}$	$-\frac{5}{13}$
$Z = 2700$		Z_j	**5**	**6**	**-1**	**-1**
		$C_j - Z_j$	0	0	1	1

Since all $C_j - Z_j \geq 0$

Hence, optimal solution is arrived with value of variables as :

$x_1 = \frac{4500}{13}, x_2 = \frac{2100}{13}$

Min $Z = 2700$

Example-5

Min $Z = 9x_1 + 10x_2$

subject to

$x_1 + 2x_2 \geq 25$

$4x_1 + 3x_2 \geq 24$

$3x_1 + 2x_2 \geq 60$

and $x_1, x_2 \geq 0$;

After introducing surplus,artificial variables

Min $Z = 9x_1 + 10x_2 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2 + MA_3$

subject to

$x_1 + 2x_2 - S_1 + A_1 = 25$

$4x_1 + 3x_2 - S_2 + A_2 = 24$

$3x_1 + 2x_2 - S_3 + A_3 = 60$

and $x_1, x_2, S_1, S_2, S_3, A_1, A_2, A_3 \geq 0$

Iteration-1		C_j	9	10	0	0	0	M	M	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	**MinRatio** $\frac{X_B}{x_1}$
A_1	M	25	1	2	-1	0	0	1	0	0	$\frac{25}{1} = 25$
A_2	M	24	(4)	3	0	-1	0	0	1	0	$\frac{24}{4} = 6 \rightarrow$
A_3	M	60	3	2	0	0	-1	0	0	1	$\frac{60}{3} = 20$
$Z = 109M$		Z_j	**$8M$**	**$7M$**	**$-M$**	**$-M$**	**$-M$**	**M**	**M**	**M**	
		$C_j - Z_j$	$-8M + 9 \uparrow$	$-7M + 10$	M	M	M	0	0	0	

Negative minimum $C_j - Z_j$ is $-8M + 9$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 6 and its row index is 2. So, the leaving basis variable is A_2.

$\therefore$ The pivot element is 4.

Iteration-2		C_j	9	10	0	0	0	M	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_3	MinRatio $\frac{X_B}{S_2}$
A_1	M	19	0	$\frac{5}{4}$	-1	$\frac{1}{4}$	0	1	0	$\frac{19}{\frac{1}{4}}=76$
x_1	9	6	1	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	0	0	---
A_3	M	42	0	$-\frac{1}{4}$	0	$\left(\frac{3}{4}\right)$	-1	0	1	$\frac{42}{\frac{3}{4}}=56 \rightarrow$
$Z=61M+54$		Z_j	9	$M+\frac{27}{4}$	$-M$	$M-\frac{9}{4}$	$-M$	M	M	
		C_j-Z_j	0	$-M+\frac{13}{4}$	M	$-M+\frac{9}{4}\uparrow$	M	0	0	

Negative minimum $C_j - Z_j$ is $-M+\frac{9}{4}$ and its column index is 4. So, the entering variable is S_2.

Minimum ratio is 56 and its row index is 3. So, the leaving basis variable is A_3.

$\therefore$ The pivot element is $\frac{3}{4}$.

Iteration-3		C_j	9	10	0	0	0	M	
B	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	MinRatio $\frac{X_B}{x_2}$
A_1	M	5	0	$\left(\frac{4}{3}\right)$	-1	0	$\frac{1}{3}$	1	$\frac{5}{\frac{4}{3}}=\frac{15}{4}=3.75 \rightarrow$
x_1	9	20	1	$\frac{2}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{20}{\frac{2}{3}}=30$
S_2	0	56	0	$-\frac{1}{3}$	0	1	$-\frac{4}{3}$	0	---
$Z=5M+180$		Z_j	9	$\frac{4M}{3}+6$	$-M$	0	$\frac{M}{3}-3$	M	
		C_j-Z_j	0	$-\frac{4M}{3}+4\uparrow$	M	0	$-\frac{M}{3}+3$	0	

Negative minimum $C_j - Z_j$ is $-\frac{4M}{3}+4$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 3.75 and its row index is 1. So, the leaving basis variable is A_1.

$\therefore$ The pivot element is $\frac{4}{3}$.

Iteration-4		C_j	9	10	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**
x_2	10	$\frac{15}{4}$	0	1	$-\frac{3}{4}$	0	$\frac{1}{4}$
x_1	9	$\frac{35}{2}$	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
S_2	0	$\frac{229}{4}$	0	0	$-\frac{1}{4}$	1	$-\frac{5}{4}$
$Z = 195$		Z_j	**9**	**10**	**-3**	**0**	**-2**
		$C_j - Z_j$	0	0	3	0	2

Since all $C_j - Z_j \geq 0$

Hence, optimal solution is arrived with value of variables as :

$x_1 = \frac{35}{2}, x_2 = \frac{15}{4}$

Min $Z = 195$

Example-6

Min $Z = 20x_1 + 24x_2 + 18x_3$

subject to

$$2x_1 + x_2 + x_3 \geq 30$$
$$x_1 + x_2 + x_3 \geq 20$$
$$x_1 + 2x_2 + x_3 \geq 24$$

and $x_1, x_2, x_3 \geq 0$;

After introducing surplus,artificial variables

Min $Z = 20x_1 + 24x_2 + 18x_3 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2 + MA_3$

subject to

$$2x_1 + x_2 + x_3 - S_1 + A_1 = 30$$
$$x_1 + x_2 + x_3 - S_2 + A_2 = 20$$
$$x_1 + 2x_2 + x_3 - S_3 + A_3 = 24$$

and $x_1, x_2, x_3, S_1, S_2, S_3, A_1, A_2, A_3 \geq 0$

Iteration-1		C_j	20	24	18	0	0	0	M	M	M	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	A_2	A_3	**MinRatio** $\frac{X_B}{x_1}$
A_1	M	30	(2)	1	1	-1	0	0	1	0	0	$\frac{30}{2} = 15 \rightarrow$
A_2	M	20	1	1	1	0	-1	0	0	1	0	$\frac{20}{1} = 20$
A_3	M	24	1	2	1	0	0	-1	0	0	1	$\frac{24}{1} = 24$
$Z = 74M$		Z_j	$4M$	$4M$	$3M$	$-M$	$-M$	$-M$	M	M	M	
		$C_j - Z_j$	$-4M + 20 \uparrow$	$-4M + 24$	$-3M + 18$	M	M	M	0	0	0	

Negative minimum $C_j - Z_j$ is $-4M + 20$ and its column index is 1. So, the entering variable is x_1.

Minimum ratio is 15 and its row index is 1. So, the leaving basis variable is A_1.

∴ The pivot element is 2.

Iteration-2		C_j	20	24	18	0	0	0	M	M	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_2	A_3	**MinRatio** $\frac{X_B}{x_2}$
x_1	20	15	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	$\frac{15}{\frac{1}{2}} = 30$
A_2	M	5	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	1	0	$\frac{5}{\frac{1}{2}} = 10$
A_3	M	9	0	$\left(\frac{3}{2}\right)$	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	0	1	$\frac{9}{\frac{3}{2}} = 6 \rightarrow$
$Z = 14M + 300$		Z_j	20	$2M + 10$	$M + 10$	$M - 10$	$-M$	$-M$	M	M	
		$C_j - Z_j$	0	$-2M + 14 \uparrow$	$-M + 8$	$-M + 10$	M	M	0	0	

Negative minimum $C_j - Z_j$ is $-2M + 14$ and its column index is 2. So, the entering variable is x_2.

Minimum ratio is 6 and its row index is 3. So, the leaving basis variable is A_3.

∴ The pivot element is $\frac{3}{2}$.

Iteration-3		C_j	20	24	18	0	0	0	M	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_2	MinRatio $\frac{X_B}{x_3}$
x_1	20	12	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	$\frac{12}{\frac{1}{3}} = 36$
A_2	M	2	0	0	$\left(\frac{1}{3}\right)$	$\frac{1}{3}$	-1	$\frac{1}{3}$	1	$\frac{2}{\frac{1}{3}} = 6 \rightarrow$
x_2	24	6	0	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{2}{3}$	0	$\frac{6}{\frac{1}{3}} = 18$
$Z = 2M + 384$		Z_j	20	24	$\frac{M}{3} + \frac{44}{3}$	$\frac{M}{3} - \frac{16}{3}$	$-M$	$\frac{M}{3} - \frac{28}{3}$	M	
		$C_j - Z_j$	0	0	$-\frac{M}{3} + \frac{10}{3} \uparrow$	$-\frac{M}{3} + \frac{16}{3}$	M	$-\frac{M}{3} + \frac{28}{3}$	0	

Negative minimum $C_j - Z_j$ is $-\frac{M}{3} + \frac{10}{3}$ and its column index is 3. So, the entering variable is x_3.

Minimum ratio is 6 and its row index is 2. So, the leaving basis variable is A_2.

$\therefore$ The pivot element is $\frac{1}{3}$.

Iteration-4		C_j	20	24	18	0	0	0
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_1	20	10	1	0	0	-1	1	0
x_3	18	6	0	0	1	1	-3	1
x_2	24	4	0	1	0	0	1	-1
$Z = 404$		Z_j	20	24	18	-2	-10	-6
		$C_j - Z_j$	0	0	0	2	10	6

Since all $C_j - Z_j \geq 0$

Hence, optimal solution is arrived with value of variables as :
$x_1 = 10, x_2 = 4, x_3 = 6$

Min $Z = 404$

Dual - Primal problems

The general linear programming problem has the form

maximize Z = $c_1x_1 + c_2x_2 + .. + c_nx_n$.

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1.$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = \leq b_m.$$
$$x_1, x_2, \ldots, x_n \geq 0.$$

Then its dual is

Minimise W = $b_1y_1 + b_2y_2 + .. + b_my_m$.

Subject to

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2$$
$$\ldots$$
$$a_{1n}y_1 + a_{2n}y_2 + .. + a_{nm}y_m \geq c_m.$$
$$y_1, y_2, \ldots, y_m \geq 0.$$

Comparing the two problem we have the following points

1. If the primal contains n variables and m constraints, the dual will contains m variables and n constraints.
2. The maximization problem in the primal become the minimization problem in the dual and vice versa.
3. The maximization problem has (≤) constraints while the minimization problem has (≥) constraints.
4. The constants c_1, c_2, …, c_n in the objective function of the primal appear in the constraints of the duals.
5. The constants b_1, b_2, …, b_n in the constraints of the primal appear in the objective function of the duals.
6. The variables in both problem are non negative.

The constraint relationship of the primal and dual can be represented in a single table as shown below

	x_1	x_2	x_3	…	x_n	
y_1	a_{11}	a_{12}	a_{13}	…	a_{1n}	$\leq b_1$
y_2	a_{21}	a_{22}	a_{23}	…	a_{2n}	$\leq b_2$
y_3	a_{31}	a_{32}	a_{33}	…	A_{3n}	$\leq b_3$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
y_m	a_{m1}	a_{m2}	a_{m3}	…	a_{mn}	$\leq b_m$
	$\geq c_1$	$\geq c_2$	$\geq c_3$	…	$\geq c_n$	

Example-1

Primal	Dual
Maximize Z= $3x_1+5x_2$ bject to $2x_1+6x_2 \leq 50$ $3x_1+2x_2 \leq 35$ $5x_1-3x_2 \leq 10.$ $x_1, x_2, \geq 0.$	Minimize W= $50y_1+35y_2+10y_3$ ect to $2y_1+3y_2+5y_3 \geq 3$ $6y_1+2y_2-3y_3 \geq 5$ $y_1, y_2, y_3, \geq 0.$

Example-2

Primal		Dual
Max Z= $3x_1+5x_2$	Max Z= $3x_1+5x_2$	Min $W=50y_1+35y_2+10y_3-2y_4+20y_5-20y_6$
t to	t to	Subject to
$2x_1+6x_2 \le 50$	$2x_1+6x_2 \le 50$	$2y_1+3y_2+5y_3-y_4+5y_5-5y_6 \ge 3$
$3x_1+2x_2 \le 35$	$3x_1+2x_2 \le 35$	$6y_1+2y_2-3y_3+0y_4+6y_5-6y_6 \ge 5$
$5x_1-3x_2 \le 10$.	$5x_1-3x_2 \le 10$.	$y_j \ge 0$.
$x_1 \ge 2$	$-x_1+0x_2 \le -2$	
$5x_1+6x_2=20$	$5x_1+6x_2 \le 20$	
	$-5x_1-6x_2 \le -20$	
$x_1, x_2, \ge 0$.	$x_1, x_2, \ge 0$.	

Example-3

Primal	Dual	Dual of Dual (=Primal)
Min Z= $3x_1+5x_2$	**Max W= $36y_1+24y_2$**	Min Q= $3u_1+5u_2$
t to	**t to**	t to
$2x_1+6x_2 \ge 36$	**$2y_1+3y_2 \le 3$**	$2u_1+6u_2 \ge 36$
$3x_1+2x_2 \ge 24$	**$6y_1+2y_2 \le 5$**	$3u_1+2u_2 \ge 24$
$x_1, x_2, \ge 0$.	**$y_1, y_2 \ge 0$.**	$u_1, u_2, \ge 0$.

Duality General Definition

We have the general LP Problem as

Minimize $f(\mathbf{X}) = CX$ (1)

Subjected to $AX \ge B$ (2)

$X \ge 0$ (3)

Where A is an mxn matrix. X and B are column n vector and X is a raw n vector. Then it's Duel is written as

Maximize $\phi(\mathbf{Y}) = B'Y$ (4)

Subjected to $A'Y \ge C'$ (5)

$Y \ge 0$ (6)

Where Y is column m vector.

Duality Theorems

Thoerem-1 The dual of the dual is the primal

Thoerem-2 (Weak Duality Theorem)**:** The value of the objective function f(X) for any feasible solution of the primal is not less than the value of the objective function $\phi(\mathbf{Y})$ for any feasible solution of the dual. (ie. To prove min $f(X) \ge \max \phi(\mathbf{Y})$)

Corollary-1: It immediately follow from equation (4) that

$$\min f(X) \ge \max \phi(\mathbf{Y})$$

Thoerem-3 (Optimality Criterion Theorem)**:** The optimum value of the primal f(X) if exist, is equal to the optimum value of the dual $\phi(\mathbf{Y})$.

Theorem 4: If the primal problem is feasible, then it has an unbounded optimum if and only if the dual has no feasible solution and vice versa.

Theorem 5 (Main Duality Theorem): If both the primal and the dual problems are feasible, then they both have optimal solutions such that their optimal values of the objective functions are equal.

Theorem 6 (Complementary Slackness Theorem**):** If in the optimum solutions of the primal and dual,

1. a primal variable is positive, then the corresponding dual slack (surplus) variable is zero.
2. If a primal slack (surplus) variable is positive, then the corresponding dual variable is zero and vice versa.

Note: This theorem is helpful in determining the optimal solution of the primal from the optimum solution of the dual or vice versa.

Possibilities for		Primal Solution		
		Infeasible	Unbounded	Optimum
Dual Solution	Infeasible	Yes	Yes	No
	Unbounded	Yes	No	No
	Optimum	No	No	Yes

The following is an example how we can determine the optimal solution of primal using the optimal solution of dual

Kuhn-Tucker Condition:

Theorem 7: X is an optimal solution for the problem

Minimize f(**X**) = CX
Subjected to AX=B
X≥0

if and only if there exist **w** and **r** such that

1. **AX=B, X≥0** (Primal feasibility)
2. $\boldsymbol{A'w + r = c}$, **r≥0** (Dual feasibility)
3. $\boldsymbol{r'x = 0}$ (Complementary Slackness)

Example: Consider the problem

Maximise $f = 3x_1 + 2x_2 + x_3 + 4x_4$,
Subject to

$$2x_1 + 2x_2 + x_3 + 3x_4 \leq 20,$$
$$3x_1 + x_2 + x_3 + 2x_4 \leq 20,$$
$$x_1, x_2, x_3, x_4 \geq 0,$$

Its dual is
Minimize $\phi = 20y_1 + 20y_2$
Subject to

$$2y_1 + 3y_2 \geq 3,$$
$$2y_1 + y_2 \geq 2,$$
$$y_1 + 2y_2 \geq 1,$$
$$3y_1 + 2y_2 \geq 4,$$
$$y_1, y_2 \geq 0.$$

This is a two variable problem whose solution can be obtained geometrically as

$$y_1 = 1.2,\ \ y_2 = 0.2,\ \phi = 28$$

After introducing the slack variables, the primal and dual constraints are

$$\begin{aligned}
&2x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 20,\\
&3x_1 + x_2 + 2x_3 + 2x_4 + x_6 = 20,\\
&2y_1 + 3y_2 - y_3 = 3,\\
&2y_1 + y_2 - y_4 = 2,\\
&y_1 + 2y_2 - y_5 = 1,\\
&3y_1 + 2y_2 - y_6 = 4,\\
&x_1, x_2, \ldots x_6,\ \ y_1, y_2, \ldots y_6 \geq 0,
\end{aligned}$$

Substituting the optimal values of $y_1 = 1.2$ and $y_2 = 1.2$ in the dual constraints, it follows that the slack variables
$y_3 = y_6 = 0$, $y_4 > 0$, $y_5 > 0$.
Thus the second and third constraints are satisfied as strict inequalities, and so the corresponding primal variables should be zero, that is, $x_2 = 0$, $x_3 = 0$. Also since the dual variables, $y_1 > 0$, $y_2 > 0$, it follows that the corresponding primal constraints should be zero, that is, $x_5 = x_6 = 0$. The primal constraints thus reduce to
$2x_1 + 3x_4 = 20$,
$3x_1 + 2x_4 = 20$,
Which give $x_1 = 4$, $x_4 = 4$. Thus optimal solution of the primal is therefore
$x_1 = x_4 = 4$, $x_2 = x_3 = 0$, $f = 28$

Application of Duality

We can find the optimal solution of the primal from the optimum solution of the dual problem. This co-existence of the solutions of the primal-dual pair is helpful in many practical situations. The following are the instances of the application of this result.

1. Usually in a l.p.p numerical work increases more with the number of constraints than the number of variables. So, if the number of constraints in primal problem is considerably larger than number of variables in it, then we can solve the dual problem with a smaller number of constraints. This approach is more economical than solving the primal problem.

2. In some situations, the dual problem can eliminate the use of artificial variables and hence the two-phase method. If the phase 1 of the two-phase method fails to eliminate the artificial variable from the basis, then we cannot move to the phase 2 and in such situation, we cannot find the optimum solution. In such situations we can use the dual problem to find the optimum solution.
3. Using the primal-dual relationship we can define a modified simplex method called dual simplex method, in which we start the iterations with an infeasible basic solution of the primal under certain conditions and proceed the iterations which will leads to the optimum solution of the primal.

This method is widely used when we have to enter a new constraint after solving the given problem. So, this method is more economical since it avoids the solution of the problems from the very beginning.

SENSITIVITY ANALYSIS

The optimal solution of the L.P.P depends on the values of $c_j \in C$, $a_{ij} \in A$, $b_i \in B$. So, these values are called coefficients of an l.p.p. In real life problems the values of these coefficients are seldom known with certainty because many of them are functions of some uncontrolled factors. So, each variation in the values of the data coefficients changes the problem, which may affect the optimal solution of the problem. However, it is not always necessary to solve the whole problem afresh to determine the new optimal solution. For this we use sensitivity analysis.

Sensitivity analysis is a systematic study of how sensitive solutions are to (small) changes in the data. The basic idea is to be able to give answers to questions of the form:
1. If the objective function changes, how does the solution change?
2. If resources available change, how does the solution change?
3. If a constraint is added to the problem, how does the solution change?

The sensitivity analysis is used to find the optimal solution of an L.P.P when changes are made in the initial system, without solving the modified problem from the very beginning. That is the sensitivity analysis is the effect of changes in the input data on the final optimal results. In sensitivity analysis we are going to study what will be the situation if any of the following 7 cases occurs one by one

1. Changes in c_j
2. Changes in b_i
3. Changes in a_{ij}
4. Introduction of new variables
5. Introduction of new constraints
6. Deletion of certain variables
7. Deletion of some constraints.

Changing Objective Function (i.e. Changes in c_j)

First up all consider the example. A company plans to produce three product A,B and C. The unit profits on these products are Rs.2/-, Rs3/- and Rs1/- respectively. The following problem gives the constrains on Labour and material. Let x_1, x_2 and x_3 are the number of units to be produced of the products A, B and C. Then the related LPP is

$$\text{Maximize } Z = 2x_1 + 3x_2 + x_3$$

Subjected to

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \leq 1$$

$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \leq 1$$

$$x_1, x_2, x_3 \geq 0$$

Now we solve the problem by the simplex method

Iteration-1

B	Cj		2	3	1	0	0	**Min Ratio** X_B/x_1
	C_B	X_B	x_1	x_2	x_3	x_4	x_5	
x_4	0	1	1/3	1/3	1/3	1	0	1/(1/3)=3
x_5	0	1	1/3	4/3	7/3	0	1	3/(4/3)=2.25→
		Zj	0	0	0	0	0	
Z=0	RC = C_j -Zj		2	3↑	1	0	0	

Iteration-2

B	Cj		2	3	1	0	0	**Min Ratio** X_B/x_1
	C_B	X_B	x_1	x_2	x_3	x_4	x_5	
x_4	0	1/4	1/4	0	-1/4	1	-1/4	(¼)/(¼)=1→
x_2	3	9/4	1/4	1	7/4	0	3/4	(9/4)/(¼)=9
		Zj	3/4	3	21/4	0	9/4	
Z=27/4	RC = C_j -Zj		5/4↑	0	-17/4	0	-9/4	

Iteration-3

B	Cj		2	3	1	0	0
	C_B	X_B	x_1	x_2	x_3	x_4	x_5
x_1	2	1	1	0	-1	4	-1
x_2	3	2	0	1	2	-1	1
		Zj	2	3	4	5	1
Z=8	RC = C_j -Zj		0	0	-3	-5	-1

Since all RC = C_j -Zj≤ 0 optimal solution is arrived and the optimum solution is X_1=1, X_2= 2 and X_3=0 with Z=8

Initial and Final Table

CB	Cj		2	3	1	0	0
	B	X_B	x_1	x_2	x_3	x_4	x_5
0	x_4	2	1/3	1/3	1/3	1	0
0	x_5	1	1/3	4/3	7/3	0	1
Z=0		Cj-Zj	2	3	1	0	0
2	x_1	1	1	0	-1	4	-1
3	x_2	2	0	1	2	-1	1
Z=8		Cj-Zj	0	0	-3	-5	-1

Suppose that you solve an LP and then wish to solve another problem with the same constraints but a slightly different objective function. When you change the objective function it turns out that there are two cases to consider. The first case is the change in a non-basic variable (a variable that takes on the value zero in the solution). In the example, the relevant non-basic variables is x_3.

What happens to your solution if the coefficient of a non-basic variable decreases? Or what is the limit in which the coefficient of a non-basic variable lies so that the optimal solution remains unaltered.

If you lower the objective function coefficient of a non-basic variable, then the solution does not change.
What if you raise the coefficient? Intuitively, raising it just a little bit should not matter, but raising the coefficient lot might induce you to change the value of x in a way that makes $x_3 > 0$. So, for a non-basic variable, you should expect a solution to continue to be valid for a range of values for coefficients of non-
basic variables. The range should include all lower values for the coefficient and some higher values. If the coefficient increases enough (and putting the variable into the basis is feasible), then the solution changes.

In the above example for the non-basic variable x_3 in the above problem we have $C_B =$ (2,3) So

$$\bar{c}_3 = c_3 - C'_B P_3 = c_3 - (2 \quad 3)\begin{pmatrix}-1\\2\end{pmatrix} = c_3 - 4$$

For optimality $c_3 - 4 \leq 0$. That is, $c_3 \leq 4$. So, as long as the unit profit on the product C is less than Rs.4/-, it is not economical to produce product C. if the profit change to a value more than 4 say 6 then the optimal solution obtained above is no more reliable and we have to redo the analysis with new cost starting from the last optimal table.

Suppose unit profit on product C is increases to Rs. 6/-. Then $\bar{c}_3$=+2 so current product mix is not optimal. So we have to do another iteration to get the optimal table. Here new objective function is
$Z=2x_1+3x_2+6x_3$

CB	Cj		2	3	6	0	0	Min Ratio X_B/x_2
	B	XB	x_1	x_2	x_3	x_4	x_5	
2	x_1	1	1	0	-1	4	-1	-1
3	x_2	2	0	1	2	-1	1	2/2=1→
		Zj	2	3	4	5	1	
Z=8		Cj-Zj	0	0	2↑	-5	-1	

CB	Cj		2	3	6	0	0
	B	XB	x_1	x_2	x_3	x_4	x_5
2	x_1	2	1	1/2	0	7/2	-1/2
6	x_3	1	0	1/2	1	-1/2	1/2
		Zj	0	2	4	5	1
Z=10		Cj-Zj	0	-1	0	-4	-2

Since all RC = C_j -Zj$\leq$ 0 optimal solution is arrived and the optimum solution is $X_1=2$, $X_2=0$, $X_3=1$ and Z=10

What happens to your solution if the coefficient of a basic variable (like x_1 or x_2 in the example) Changes? In this case for all non basic variable, first calculate for all $\bar{c}_j = c_j - C'_B P_j$.
If all $\bar{c}_j \leq 0$ the present solution holds
If any of $\bar{c}_j > 0$ the present solution is not optimal
Find the new solution from the last iteration table with new $c_j{}'s$

Adding a Constraint

If you add a constraint to a problem, two things can happen. Your original solution satisfies the constraint or it doesn't. If it does, then you are solution holds. If you had a solution before and then the solution is still feasible for the new problem. If the original solution does not satisfy the new constraint, then possibly the new problem is infeasible. If not, then there is another solution and we have to redo the analysis with new constrains starting from the last optimal table.

Changing a Right-Hand Side Constant (change in b_i's)

When you changed the amount of resource in a non-binding constraint, then increases never changed your solution. Small decreases also did not change anything, but if you decreased the amount of resource enough to make the constraint binding, your solution could change. Note the similarity between this analysis and the case of changing the coefficient of a non-basic variable in the objective function.

Changes in the right-hand side of binding constraints always change the solution (the value of x must adjust to the new constraints). We saw earlier that the dual variable associated with the constraint measures how much the objective function will be influenced by the change.

Suppose in the above problem the labour is increased by 1 unit. So the new RHS is 2 for the first constraint. Then the new solution is $x_1=5$, $x_2=1$ and Z=Rs.13. So the profit is increased by Rs. 5 (13-8). Now let the extra unit of labour costs Rs.4/-. Now the question is whether it is profitable to employee another labour. For this we compare the increased profit by added cost of employing a labour. Which gives us the value of Rs.1/-. So we can conclude that it is profitable to employee an additional labour. So this leads to the concept of shadow price. The increase profit (Rs.5/- in the previous example) per unit increase of labour availability is called shadow price for labour constraint.

Definition. The *shadow price* associated with a particular constraint is the change in the optimal value of the objective function per unit increase in the righthand-side value for that constraint, all other problem data remaining unchanged.

Shadow prices reflect the net change in the optimal value of Z per unit increase on the constraint resources, as long as the variations in the constraint resources does not change the optimal basis. Every constraints has a shadow price